2015. 05. 28. Inf. an. <u>Szigorlat</u> MINTAMEGOLDÁS | —1—

1, <u>Rendőr-elv</u>: Ha $\forall n \in \mathbb{N}$ esetén $a_n \le b_n \le c_n$, és

$\boxed{4}$ $\exists \lim\limits_{n\to\infty} a_n = A$, $\exists \lim\limits_{n\to\infty} c_n = A$, akkor $\exists \lim\limits_{n\to\infty} b_n = A$. (4)

($\forall n \in \mathbb{N}$ helyett elég, ha $\forall n > N_0 \in \mathbb{N}$ esetén teljesül a felt.)

$\boxed{8}$ <u>Bizonyítás</u>:

$a_n \longrightarrow A$, így $A-\varepsilon < a_n < A+\varepsilon$, ha $n > N_a(\varepsilon)$ $(\varepsilon>0)$

$c_n \longrightarrow A$, így $A-\varepsilon < c_n < A+\varepsilon$, ha $n > N_c(\varepsilon)$ $(\varepsilon>0)$ (4)

Így ha $n > \max\{N_a(\varepsilon), N_c(\varepsilon)\}$, akkor (4)

$A-\varepsilon < a_n \le b_n \le c_n < A+\varepsilon$, azaz $|b_n - A| < \varepsilon$, azaz $b_n \longrightarrow A$ ✓

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2, a, <u>T.</u>: Legyen $a_n > 0$.

$\boxed{4}$ i, Ha $\exists \lim\limits_{n\to\infty} \sqrt[n]{a_n} = q > 1$, akkor $\sum\limits_{n=1}^{\infty} a_n = \infty$ (divergens) (2)

ii, Ha $\exists \lim\limits_{n\to\infty} \sqrt[n]{a_n} = q < 1$, akkor $\sum\limits_{n=1}^{\infty} a_n < \infty$ (konvergens) (2)

b,

$\boxed{8}$ $$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n+2}{n+1}\right)^{2n^2+n}} = \left(\frac{n+2}{n+1}\right)^{2n+1} = \frac{n+2}{n+1}\cdot\left[\frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n}\right]^2 \longrightarrow$$

$$\longrightarrow 1\cdot\left(\frac{e^2}{e^1}\right)^2 = e^2 > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n = \infty \text{ (div.)}$$

Felhasználtuk, hogy $\left(1+\frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^x$

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3, a, <u>Átviteli elv</u>: Legyen x_0 torlódási pontja D_f-nek. Ekkor

$\boxed{4}$ $$\lim_{x\to x_0} f(x) = A \Longleftrightarrow \forall \underset{x_n \ne x_0,\ x_n \in D_f}{x_n \longrightarrow x_0} \text{ esetén } \lim_{n\to\infty} f(x_n) = A$$

$\boxed{5}$ b, $\lim\limits_{x\to\infty} x^2 \sin(2x) = ?$

$$\left.\begin{array}{l} \text{Ha } 2x_n = n\pi;\ x_n = \frac{n\pi}{2} \xrightarrow[n\to\infty]{} \infty, \text{ akkor } x_n^2 \sin(\overbrace{2x_n}^{n\pi}) = 0 \\ \text{Ha } 2y_n = \frac{\pi}{2} + 2n\pi;\ y_n = \frac{\pi}{4} + n\pi \to \infty, \text{ akkor } y_n^2 \underbrace{\sin(\overbrace{2y_n}^{\pi/2+2n\pi})}_{1} \longrightarrow \infty \end{array}\right\} \ne$$

Tehát nem létezik a határérték.

4, $\boxed{12}$ $f(x) = \ln(2+3x^2)$ $D_f = \mathbb{R}$

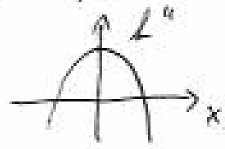
$f'(x) = \dfrac{6x}{2+3x^2}$ (2)

x	$x<0$	0	$0<x$
f'	$-$	0	$+$
f	↘	lok. min.	↗

(3)

f-nek 0-ban lokális minimuma van.

$f''(x) = \dfrac{6(2+3x^2)-(6x)^2}{(2+3x^2)^2} = \dfrac{12-18x^2}{(2+3x^2)^2} = -\dfrac{18}{(2+3x^2)^2}\left(x+\sqrt{\tfrac{2}{3}}\right)\left(x-\sqrt{\tfrac{2}{3}}\right)$ (2)



x	$x<-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{2/3}<x<\sqrt{2/3}$	$+\sqrt{\frac{2}{3}}$	$\sqrt{2/3}<x$
f''	$-$	0	$+$	0	$-$
f	$\cap$	infl.	$\cup$	infl.	$\cap$

(5)

f konvex a $\left[-\sqrt{2/3}, +\sqrt{2/3}\right]$ intervallumon.

f konkáv a $\left(-\infty, -\sqrt{2/3}\right]$ és a $\left[\sqrt{2/3}, +\infty\right)$ intervallumon.

Inflexiós pontok: $\pm\sqrt{2/3}$.

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5, a, $\boxed{5}$

$\displaystyle\int \underset{u}{(2x+1)} \underset{v'}{\sin(5x)}\,dx = -(2x+1)\frac{\cos(5x)}{5} + \int 2\cdot\frac{\cos(5x)}{5}\,dx =$ (3)

$u' = 2 \quad v = -\dfrac{\cos(5x)}{5}$ $\Big| = \underline{\underline{-\dfrac{1}{5}(2x+1)\cos(5x) + \dfrac{2}{25}\sin(5x) + C}}$ (2)

b, $\boxed{5}$ $\displaystyle\int_{x=0}^{\pi} \sin^3 x\,dx = \int_{x=0}^{\pi} \sin x\,(1-\cos^2 x)\,dx = \int_0^{\pi} \sin x\,dx - \int_0^{\pi} \sin x \cos^2 x\,dx =$ (2)

$= \Big[-\cos x\Big]_0^{\pi} + \Big[\dfrac{1}{3}\cos^3 x\Big]_0^{\pi} = \big(+1-(-1)\big) + \dfrac{1}{3}\big(-1-1\big) = 2 - \dfrac{2}{3} = \underline{\underline{\dfrac{4}{3}}}$ (3)

c, $\boxed{5}$ $x^2+7x+6 = (x+1)(x+6)$; $\dfrac{1}{x^2+7x+6} = \dfrac{A}{x+1} + \dfrac{B}{x+6}$ (2)

$1 = A(x+6) + B(x+1)$ $\quad x=-6:\ 1 = -5B \Rightarrow B = -\frac{1}{5}$

$\quad x=-1:\ 1 = 5A \Rightarrow A = 1/5$ (2)

$\displaystyle\int \frac{1}{x^2+7x+6}\,dx = \frac{1}{5}\int\frac{1}{x+1}\,dx - \frac{1}{5}\int\frac{1}{x+6}\,dx = \underline{\underline{\frac{1}{5}\ln|x+1| - \frac{1}{5}\ln|x+6| + C}}$ (1)

6, [7] $y''(x)+6y'(x)+Cy(x)=5e^{-2x}$

4 Akkor van külső rezonancia, ha a -2 gyöke a $\lambda^2+6\lambda+C$ polinomnak, azaz $(-2)^2+6\cdot(-2)+C=0$
$\Longrightarrow C=-4+12=\underline{\underline{8}}$

3 Ekkor $\lambda^2+6\lambda+8=(\lambda+2)(\lambda+4) \Longrightarrow \lambda_1=-2;\ \lambda_2=-4$
és $\lambda=-2$ egyszeres gyök, (1) tehát $y_{i,p}(x)=Axe^{-2x}$ (2) alakban keresendő.

7*, [10] $f(x,y)=x^3+y^3-3xy$ $\mathbb{R}^2$-en akárhányszor diff. ható. (polinom)

[4] Szükséges felt.:

$$\operatorname{grad} f(x,y)=\begin{bmatrix} f'_x \\ f'_y \end{bmatrix}=\begin{bmatrix} 3x^2-3y \\ 3y^2-3x \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{matrix} 3x^2-3y=0 \\ 3y^2-3x=0 \end{matrix} \quad (2)$$

$$\left.\begin{matrix} y=x^2 \\ x=y^2 \end{matrix}\right\} \begin{matrix} x=x^4 \\ x(x^3-1)=0 \end{matrix} \Longrightarrow \left.\begin{matrix} \underline{x_1=0,\ y_1=0} \\ \underline{x_2=1,\ y_2=1} \end{matrix}\right\} (2)$$

$$\underline{\underline{H}}(x,y)=\begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{bmatrix}=\begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

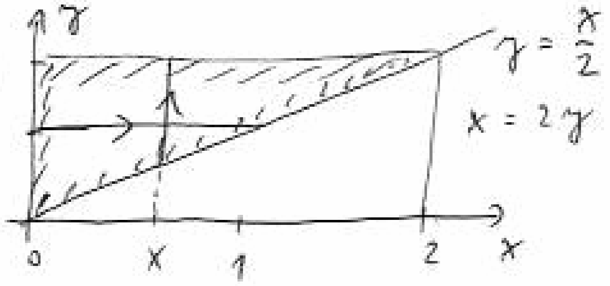
[3] $\underline{(0,0)\text{-ban:}}$ $\det(\underline{\underline{H}}(0,0))=\begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix}=-9<0 \Longrightarrow$ nincs lokális szélsőérték.

[3] $\underline{(1,1)\text{-ben:}}$ $\det(\underline{\underline{H}}(1,1))=\begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix}=36-9=27>0 \Longrightarrow \exists$ lok. szélsőérték

$f''_{xx}(1,1)=6>0 \Longrightarrow$ <u>lokális minimum van (1,1)-ben</u>

8* $\boxed{8}$

$$\int_{x=0}^{2}\left(\int_{y=x/2}^{1} e^{y^2}\,dy\right)dx =$$



$$= \int_{y=0}^{1}\left(\int_{x=0}^{2y} e^{y^2}\,dx\right)dy \overset{4}{=}$$

$$= \int_{y=0}^{1} e^{y^2}(2y-0)\,dy \overset{2}{=} \left[e^{y^2}\right]_0^1 = \underline{\underline{e-1}} \;\; 2$$

$e^{f}\cdot f'\,dt$

9* $\boxed{7}$ a, $(f*g)(x) = \int_{t=-\infty}^{\infty} f(t)\,g(\underbrace{x-t})\,dt \overset{3}{=} \int_{y=+\infty}^{-\infty} f(x-y)\,g(y)\,(-dy) =$

$y = x-t\;;\; t = x-y$

$dy = -dt$

$$= \int_{y=-\infty}^{\infty} g(y)\,f(x-y)\,dy = (g*f)(x) \;\; 4$$

$\boxed{8}$ b, $\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx \overset{2}{=} \int_0^{\infty} e^{-i\omega x}\, e^{-x}\,dx =$

$$= \int_0^{\infty} e^{-(i\omega+1)x}\,dx \overset{2}{=} \lim_{R\to\infty}\left[\frac{e^{-(i\omega+1)x}}{-(i\omega+1)}\right]_{x=0}^{R} =$$

$$= \lim_{R\to\infty}\Big(\underbrace{e^{-(i\omega+1)R}}_{e^{-i\omega R}\cdot e^{-R}} - 1\Big)\cdot\frac{-1}{1+i\omega} = \underline{\underline{\frac{1}{1+i\omega}}} \overset{4}{=} \underline{\underline{\frac{1-i\omega}{1+\omega^2}}}$$

$e^{-i\omega R}$: abs. korl.

$e^{-R} \xrightarrow{R\to\infty} 0$

Ha nincs kiírva a $\lim_{R\to\infty}$, akkor 1 pont levonás.