1, a, $\boxed{5p}$

$$\lim_{x\to 0} \underbrace{(3^x-1)}_{\downarrow 0 \ (1)} \cdot \underbrace{\mathrm{th}\left(\frac{1}{x}\right)}_{\text{korlátos } (2)} = \underline{\underline{0}} \quad (2)$$

b, $\boxed{6p}$

$$\lim_{x\to\infty} \frac{\ln(x^3+3x) \to \infty}{\sqrt[5]{x} \to \infty} \ (1) \ \overset{L'H}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{\left(\frac{3x^2+3}{x^3+3x}\right)}{\frac{1}{5}x^{-4/5}} \ (3) =$$

$$= \lim_{x\to\infty} 5\cdot \underbrace{x^{4/5}\cdot\frac{x^2}{x^3}}_{x^{-1/5}\to 0} \cdot \underbrace{\frac{3+3/x^2}{1+3/x^2}}_{\to 3} = \underline{\underline{0}} \quad (2)$$

($3/x^2 \to 0$)

c, $\boxed{8p}$

$$\lim_{x\to 0-0}\left(\frac{1}{\mathrm{sh}\,x}-\frac{1}{x}\right) = \lim_{x\to 0-0} \frac{x-\mathrm{sh}\,x \to 0}{x\,\mathrm{sh}\,x \to 0} \ (2p) \ \overset{L'H}{=} \lim_{x\to 0-0}\frac{1-\mathrm{ch}\,x}{\mathrm{sh}\,x + x\,\mathrm{ch}\,x} \ (3) =$$

$$\overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0-0} \frac{-\mathrm{sh}\,x}{2\,\mathrm{ch}\,x + x\,\mathrm{sh}\,x} \ (2) = \frac{0}{2} = \underline{\underline{0}} \quad (1)$$

d, $\boxed{8p}$

$$\lim_{x\to\frac{\pi}{2}-0} (\sin x)^{\mathrm{tg}\,x} = \lim_{x\to\frac{\pi}{2}-0} e^{\mathrm{tg}\,x\cdot\ln(\sin x)} \ (2)$$

; a kitevő:

$$\lim_{x\to\frac{\pi}{2}-0} \mathrm{tg}\,x\,\ln(\sin x) = \lim_{x\to\frac{\pi}{2}-0}\frac{\ln(\sin x)\to 0}{\mathrm{ctg}\,x \to 0} \ (2) \ \overset{L'H}{=} \lim_{x\to\frac{\pi}{2}-0}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{-1}{\sin^2 x}\right)} \ (2) =$$

$$= \lim_{x\to\frac{\pi}{2}-0}(-\sin x\cdot\cos x) = 0 \quad (1)$$

Tehát $\lim_{x\to\frac{\pi}{2}-0} (\sin x)^{\mathrm{tg}\,x} = e^0 = \underline{\underline{1}}$ (1)

2, $\boxed{8}$ $\lim\limits_{x\to 4}\sqrt{2x+1}=3$ Legyen $\varepsilon>0$.

$$|\sqrt{2x+1}-3| = \left|\frac{2x+1-9}{\sqrt{2x+1}+3}\right| = \frac{2\,|x-4|}{\sqrt{2x+1}+3} \le \frac{2}{3}|x-4| < \varepsilon,$$ (3) (3)

ha $|x-4|<\frac{3\varepsilon}{2}$, tehát $\delta(\varepsilon)=\frac{3\varepsilon}{2}$ (2)

3, $\boxed{13p}$

$$f(x)=\frac{|x^3+4x^2+4x|}{x^3-4x}=\frac{|x|\cdot(x+2)^2}{x\,\cancel{(x+2)}(x-2)}\quad;\quad D_f=\mathbb{R}\setminus\{0,+2,-2\}$$

A nevező zérushelyeinél szakad a fgv., azaz $0,\ \pm 2$-ben. (2)

$x_1=0$-ban:

$$\lim_{x\to 0\pm 0} f(x)=\underbrace{\frac{(0+2)^2}{(0+2)(0-2)}}_{-1}\cdot\underbrace{\lim_{x\to 0\pm 0}\frac{|x|}{x}}_{\pm 1}=\mp 1;$$ első fajú, véges ugrás (4)

$x_2=+2$-ben:

$$\lim_{x\to +2\pm 0} f(x)=\frac{2\cdot 4}{\underbrace{2}_{>0}}\cdot\underbrace{\lim_{x\to 2\pm 0}\frac{1}{x-2}}_{\pm\infty}=\pm\infty;$$ másodfajú szakadás (3)

$x_3=-2$-ben:

$$\lim_{x\to -2\pm 0} f(x)=\underbrace{\frac{|-2|}{(-2)\cdot(-4)}}_{\frac{1}{4}}\cdot\underbrace{\lim_{x\to -2\pm 0}(x+2)}_{0}=0;$$ első fajú, megszüntethető (4)

$\boxed{-3-}$

4, $\boxed{11p}$

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x^2}\right), & \text{ha } x \neq 0 \\ 0, & \text{ha } x = 0 \end{cases}$$

Ha $x \neq 0$, akkor f folytonos, diff.-ható, és (3)

$$f'(x) = \sin\left(\frac{1}{x^2}\right) + x \cdot \cos\left(\frac{1}{x^2}\right) \cdot (-2) \cdot \frac{1}{x^3} \quad (3)$$

korl.

$x = 0$-ban is f folytonos, mert $\lim\limits_{x \to 0} \underset{\searrow 0}{x} \sin \frac{1}{x^2} = 0$ (2)

$$f'(0) = \lim_{x \to 0} \frac{x \sin(1/x^2) - 0}{x} = \lim_{x \to 0} \sin\left(\frac{1}{x^2}\right) = \nexists$$

$\left(x_k = \frac{1}{\sqrt{k\pi}} \right.$, ill. $y_k = \frac{1}{\sqrt{2k\pi + \frac{\pi}{2}}}$ -re

$x_k \xrightarrow{k \to \infty} 0$; $y_k \xrightarrow{k \to \infty} 0$, de $\sin\left(\frac{1}{x_k^2}\right) = \sin(k\pi) = 0$, és

$\left. \sin\left(\frac{1}{y_k^2}\right) = \sin\left(2k\pi + \frac{\pi}{2}\right) = 1 \right)$ (3)

5, $\boxed{18p}$

$$f(x) = 3 - 4 \arcsin(e^{5x} - 2)$$

Az $x \mapsto e^{5x} - 2$ szig. mon. növő, az

$x \mapsto -4 \arcsin x$ — " — csökkenő, tehát $f(x)$

szigorúan monoton csökkenő, így invertálható. (2)

Vagy másképp: $f'(x) = -4 \dfrac{5e^{5x}}{\sqrt{1-(e^{5x}-2)^2}} < 0$, ha $x \in \mathrm{Int}(D_f)$,

tehát f szig. mon. csökken.

$D_{\arcsin} = [-1, +1]$, így $-1 \leq e^{5x} - 2 \leq +1$

$+1 \leq e^{5x} \leq +3$

$0 \leq x \leq \dfrac{\ln 3}{5} \implies D_f = \left[0, \dfrac{\ln 3}{5}\right]$

(3)

Inverz fgv.:

$y = 3 - 4 \arcsin(e^{5x} - 2)$

$\dfrac{3-y}{4} = \arcsin(e^{5x} - 2) \implies x = \dfrac{1}{5} \ln\left(\sin\left(\dfrac{3-y}{4}\right) + 2\right)$

$$\underline{\underline{f^{-1}(x) = \frac{1}{5} \ln\left(\sin\left(\frac{3-x}{4}\right) + 2\right)}} \quad (4)$$

$R_{f^{-1}} = D_f = \underline{\underline{\left[0, \dfrac{\ln 3}{5}\right]}}$ (2) ; $D_{f^{-1}} = R_f = \left[f\left(\dfrac{\ln 3}{5}\right), f(0)\right] = \left[3 - \dfrac{4\pi}{2}, 3 + \dfrac{4\pi}{2}\right] =$

$= \underline{\underline{[3 - 2\pi, 3 + 2\pi]}}$

(3)

$$(f^{-1})'(x) = \frac{1}{5} \, \frac{\cos\left(\frac{3-x}{4}\right) \cdot \left(\frac{-1}{4}\right)}{\sin\left(\frac{3-x}{4}\right) + 2} \quad (4)$$

6, $\boxed{12}$ $f(x) = x^2 \ln x$ $\qquad D_f = (0, +\infty)$

a, $f'(x) = 2x \ln x + x^2 \cdot \frac{1}{x} = x(2\ln x + 1)$ ②

$\boxed{7}$ $f''(x) = 2\ln x + 1 + x \cdot \frac{2}{x} = 2\ln x + 3 > 0 \iff \ln x > -\frac{3}{2}$ ②

$\iff x > e^{-3/2}$

x	$0 < x < e^{-3/2}$	$e^{-3/2}$	$e^{-3/2} < x$
f''	−	0	+
f	∩	infl. pont.	∪

③

b, $f(e^{-3/2}) = e^{-3} \cdot \left(-\frac{3}{2}\right)$ ①; $f'(e^{-3/2}) = e^{-3/2} \cdot (-3+1) = -2e^{-3/2}$ ①

$\boxed{5}$ Érintő: $y - y_0 = f'(x_0) \cdot (x - x_0)$ ②, $\underline{\underline{y_e + \frac{3}{2}e^{-3} = -2e^{-3/2} \cdot (x - e^{-3/2})}}$ ①

———— ∘ ————

7, $\boxed{11}$ $f(x) = \frac{x}{x^2+4}$; $f'(x) = \frac{x^2+4-x \cdot 2x}{(x^2+4)^2} = \frac{4-x^2}{(x^2+4)^2} = \frac{(2+x)(2-x)}{(x^2+4)^2}$ ② ①

a, $(x^2+4)^2 > 0$

$\boxed{7}$ $f'(x) > 0 \iff -2 < x < +2$

x	$x < -2$	-2	$-2 < x < +2$	$+2$	$+2 < x$
f'	−	0	+	0	−
f	↘	lok. min	↗	lok. max.	↘

④

b, f folytonos, tehát Weierstrass II. tétele értelmében felveszi

$\boxed{4}$ maximumát a $[0, 4]$ intervallumon. Az int. végpontjaiban, ill. a lokális max. helyen f értéke:

$f(0) = 0$; $f(+2) = \frac{2}{4+4} = \frac{1}{4}$; $f(4) = \frac{4}{16+4} = \frac{1}{5}$

Tehát a fgv. maximuma a $[0, 4]$ int.-on: $\underline{\underline{f(2) = \frac{1}{4}}}$ ④

8,

$$f(x) = \cos(\ln(x^3+x))$$

$$f'(x) = -\sin(\ln(x^3+x)) \cdot \frac{3x^2+1}{x^3+x}$$ ⑤

$$g(x) = \frac{x\,\mathrm{sh}\,x}{e^x} \; ; \quad g'(x) = \frac{(\mathrm{sh}\,x + x\,\mathrm{ch}\,x)e^x - x\,\mathrm{sh}\,x\, e^x}{e^{2x}}$$ ⑤

9, a,

$$\lim_{x\to 0} \frac{\sin(3x^2)}{\sin(2x^2)} = \lim_{x\to 0} \underbrace{\frac{\sin(3x^2)}{3x^2}}_{\downarrow\, 1} \cdot \underbrace{\frac{2x^2}{\sin(2x^2)}}_{\downarrow\, 1} \cdot \frac{3}{2} = \frac{3}{2}$$ ⑥

(Lehet L'Hospital szabállyal is, de úgy hosszadalmas.)

b,

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{x-1}} = 1^2+1+1 = 3$$ ④