2.1 $g(z)=\frac{2}{4-2^z}$ $\quad g(0)=\frac{2}{4-1}=\frac{2}{3}$ $\quad g(1)=\frac{2}{4-2}=1$ ✓

$$g'(z)=+\frac{2}{(4-2^z)^2}(+2^z)\ln 2=2\ln 2\,\frac{2^z}{(4-2^z)^2}$$

$$g'(0)=\frac{2\ln 2}{(4-1)^2}=\frac{2\ln 2}{9}$$

$$g''(z)=2\ln 2\,\frac{2^z\ln 2\,(4-2^z)^2+2^z\,2(4-2^z)(+2^z)\ln 2}{(4-2^z)^{\cancel{4}\,3}}=2(\ln 2)^2\,2^z\,\frac{4-2^z+2\cdot 2^z}{(4-2^z)^3}$$

$$g'(1)=\frac{2\ln 2\cdot 2}{(4-2)^2}=\ln 2$$

$$g''(1)=2(\ln 2)^2\,2\,\frac{4+2}{(4-2)^3}=3\ln^2 2$$

a) $\boxed{EX=g'(1)=\ln 2}$

b) $D^2X=g''(1)+g'(1)-[g'(1)]^2=3\ln^2 2+\ln 2-\ln^2 2=2\ln^2 2+\ln 2$

$\Rightarrow \boxed{DX=\sqrt{2\ln^2 2+\ln 2}\approx 1.2861}$

c) $\boxed{P(X=0)=g(0)=\frac{2}{3}}$

$\boxed{P(X=1)=g'(0)=\frac{2}{9}\ln 2}$

2.2 I $g(z)=\frac{1}{4}+\frac{1}{4}z+\frac{1}{4}z^2+\frac{1}{4}z^3=\frac{1+z+z^2+z^3}{4}$, $\quad m=\frac{0+1+2+3}{4}=1.5$

a) $g_{Z_1}(z)=g(z)=\frac{1+z+z^2+z^3}{4}$

b) $g_2(z)=g(g(z))=\frac{1}{4}\left[1+\frac{1+z+z^2+z^3}{4}+\left(\frac{1+z+z^2+z^3}{4}\right)^2+\left(\frac{1+z+z^2+z^3}{4}\right)^3\right]$

c) $m_{10}=m^{10}=1.5^{10}\approx 57.7$

d) $\cancel{r_0=g(0)}$ $r_0=0$, $\quad r_1=g(r_0)=g(0)=\frac{1}{4}$ $\quad r_2=g(r_1)=g\left(\frac{1}{4}\right)=\frac{1}{4}\left[1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}\right]\approx 0.332$

$r_3=g(r_2)\approx g(0.332)\approx 0.3697$; $\quad r_4=g(r_3)\approx 0.3892$

$\Rightarrow \boxed{P(Z_4=0)\approx 0.3892}$

Megy

e.) $m > 1$ miatt a folyamat szuperkritikus $\Rightarrow$ a $g(z) = z$ egyenlet $[0,1)$-beli megoldását keressük:

$$\frac{1+z+z^2+z^3}{4} = z \qquad z^3 + z^2 - 3z + 1 = 0 \qquad z = 1 \text{ persze megoldás}$$

$$(z-1)(z^2+2z-1) = 0$$

$$z = 1 \qquad z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2} = \begin{cases} -1-\sqrt{2} < -1 \\ -1+\sqrt{2} = 0.414 \end{cases}$$

$$\Rightarrow \boxed{P(\text{kávéstünet}) = P(\text{kihalás}) = \sqrt{2} - 1 \approx 0.4142}$$

f.) $m > 1 \Rightarrow \boxed{EN = \infty}$

II) $g(z) = \frac{5}{10} + \frac{2}{10}z + \frac{2}{10}z^2 + \frac{1}{10}z^3 = \frac{5+2z+2z^2+z^3}{10}$ $\qquad m = \frac{5\cdot 0 + 2\cdot 1 + 2\cdot 2 + 1\cdot 3}{10} = \frac{9}{10} = 0.9$

a) $g_1(z) = g(z) = \frac{5+2z+2z^2+z^3}{10}$

b.) $g_2(z) = g(g(z)) = \frac{1}{10}\left[5 + 2\,\frac{5+2z+2z^2+z^3}{10} + 2\left(\frac{5+2z+2z^2+z^3}{10}\right)^2 + \left(\frac{5+2z+2z^2+z^3}{10}\right)^3\right]$

c.) $M_{10} = m^{10} = \left(\frac{9}{10}\right)^{10} \approx 0.35$

d.) $r_0 = 0; \quad r_1 = g(r_0) = g(0) = \frac{5}{10} \qquad r_2 = g(r_1) = g\left(\frac{5}{10}\right) = \frac{1}{10}\left[5 + 2\cdot\frac{5}{10} + 2\cdot\left(\frac{5}{10}\right)^2 + \left(\frac{5}{10}\right)^3\right] =$

$= \frac{1}{10}\left(5 + 1 + \frac{1}{2} + \frac{1}{8}\right) = \frac{6.625}{10} = 0.6625 \qquad r_3 = g(r_2) = g(0.6625) \approx 0.7494$

$r_4 = g(r_3) \approx g(0.7494) \approx 0.8043 \qquad \Rightarrow \boxed{P(Z_4 = 0) \approx 0.8043}$

e.) $m < 1$ miatt a folyamat szubkritikus $\Rightarrow$ $P(\text{kávéstünet}) = P(\text{kihalás}) = 1$

f.) $m < 1$ miatt $\boxed{EN = \frac{1}{1-m} = \frac{1}{1-\frac{9}{10}} = 10}$

Vegy

3.1

~~a)~~ $X_i := \begin{cases} 1, & \text{ha az } i\text{-edik utas nem tud úszni} \\ 0, & \text{ha tud} \end{cases}$ $i=1,2,\dots,100$

$n := 100$

Így $\mathbb{P}(X_i=0) = \mathbb{P}(X_i=1) = \frac{1}{2}$, $S_n := X_1 + \dots + X_{100}$ az úszni nem tudó utasok száma.

~~$X_i \sim B(p) \Rightarrow \mathbb{E}X_i = m = \frac{1}{2}$~~;

a.) ~~$\mathbb{P}(S_n > 75)$~~ ~~CHT~~ $X_i \sim B(p)$ $p = \frac{1}{2}$

$$\Rightarrow m = \mathbb{E}X_i = p = \frac{1}{2} \quad ; \quad \sigma = \sqrt{\operatorname{Var} X_i} = \sqrt{pq} = \frac{1}{2}$$

a.) $$\mathbb{P}(S_n > 75) = 1 - \mathbb{P}(S_n \le 75) \overset{CHT}{\approx} 1 - \phi\left(\frac{75 - n\cdot m}{\sqrt{n}\cdot\sigma}\right) = 1 - \phi\left(\frac{75-50}{10\cdot\frac{1}{2}}\right) =$$

$= 1 - \phi(5)$, ahol ϕ a standard normális eo-fv.

$\phi(5)$ már nincs benne a táblázatunkban, $\Rightarrow \phi(5) > 0.9999$

$\Rightarrow$ $\boxed{\mathbb{P}(S_n > 75) < 0.0001}$ lenne, ha a CHT becslés jó lenne.

Igazából $\phi(5) \approx 0.9999971$ $\Rightarrow$ ~~$\mathbb{P}(S_n > 7$~~ $1-\phi(5) \approx 29 \cdot 10^{-7}$

b.) $$\delta = \mathbb{E}(|X-m|^3) = \frac{1}{2}\left|1-\frac{1}{2}\right|^3 + \frac{1}{2}\left|0-\frac{1}{2}\right|^3 = \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{1}{8} = \frac{1}{8}$$

$$\Rightarrow \text{CHT becslés hibája} \underset{\text{Esseen}}{\overset{\text{Berry}}{\le}} \frac{C\,\delta}{\sqrt{n}\,\sigma^3} = \frac{0.4748\cdot 1/8}{\sqrt{100}\cdot(1/2)^3} \approx 0.047 \approx 4.7\%$$

c.) X_i korlátos: $a_i \le X_i \le b_i$, ahol $\boxed{a_i = 0, \quad b_i = 1}$ $\Longrightarrow K = \mathbb{E}S_n + t = 50 + t = 75$, vagyis $t = 25$ választással

$$\mathbb{P}(S_n > \overset{75}{K}) = \mathbb{P}(S_n > \overset{50}{\mathbb{E}S_n} + \overset{25}{t}) \le \exp\left\{-\frac{2t^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right\} = \exp\left\{-\frac{2\cdot 25^2}{100\cdot 1^2}\right\} = e^{-12.5} \approx 3.7\cdot 10^{-6}$$