MINTAMEGOLDÁS -1-

1, a, $y'(x) = f(x)\cdot y(x)$ szeparálható $y \equiv 0$ megoldás

$\boxed{7}$

$$\int \frac{dy}{y} = \int f(x)\,dx \; ; \quad \ln|y| = \int f(x)\,dx = F(x) + C$$

$$y(x) = \pm e^{C}\cdot e^{F(x)} \; ; \quad \underline{\underline{y_{H,\text{ált}}(x) = K e^{F(x)}}} \,, \text{ ahol } K \in \mathbb{R},$$

F az f primitív fv-e.

$\boxed{9}$ b, $y'(x) = \dfrac{x+2}{x^2+1}\, y(x) \; ; \quad y(0) = 2$

$$\underbrace{\int \frac{dy}{y}}_{I_1} = \underbrace{\int \frac{x+2}{x^2+1}\,dx}_{I_2} \; ① ; \qquad I_1 = \ln|y| + C \; ②$$

$$I_2 = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx + 2\int \frac{1}{x^2+1} = \frac{1}{2}\ln(1+x^2) + 2\,\text{arctg}\,x + C \; ③$$

f'/f alak

$$y_{H,\text{ált}}(x) = K\cdot \exp\left(\frac{1}{2}\ln(1+x^2) + 2\,\text{arctg}\,x\right) \; ①$$

$$y(0) = 2 \Rightarrow 2 = K\exp(0+0) = K \Rightarrow \underline{\underline{y_{\text{kezd}}(x) = 2e^{\frac{1}{2}\ln(1+x^2) + 2\,\text{arctg}\,x}}} \; ②$$

$$= \underline{\underline{2\sqrt{1+x^2}\, e^{2\,\text{arctg}\,x}}}$$

— o —

2, a, Legyen y_1 és y_2 megoldás, azaz

$\boxed{5}$

$$y_1^{(n)}(x) + \sum_{k=0}^{n-1} f_k(x)\, y_1^{(k)}(x) = 0 \qquad / \cdot \alpha \in \mathbb{R}$$

$$\oplus \quad y_2^{(n)}(x) + \sum_{k=0}^{n-1} f_k(x)\, y_2^{(k)}(x) = 0 \qquad / \cdot \beta \in \mathbb{R}$$

$$\underbrace{\alpha y_1^{(n)}(x) + \beta y_2^{(n)}(x)}_{(\alpha y_1 + \beta y_2)^{(n)}(x)} + \sum_{k=0}^{n-1} \underbrace{\alpha f_k(x)\, y_1^{(k)}(x) + \beta f_k(x)\, y_2^{(k)}(x)}_{f_k(x)(\alpha y_1 + \beta y_2)^{(k)}(x)} = 0$$

Tehát $\alpha y_1 + \beta y_2$ is megoldás. ✓

$\boxed{-2-}$

2/b, $y^{(4)}+8y''+16y=0 \Rightarrow \lambda^4+8\lambda^2+16=(\lambda^2+4)^2=(\lambda+2i)^2(\lambda-2i)^2$

$\boxed{6}$

$\lambda_{1,2}=\pm 2i$ kétszeres gyökpár (3)

$\Rightarrow \underline{y_{H,ált}(x)=(Ax+B)\sin(2x)+(Cx+D)\cos(2x)}$ (3)

3, $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(-5)^{n+2}}{5^{2n}\sqrt{n}}}_{a_n}(x+2)^n$ Hányados kritériummal:

$\boxed{8}$ $x_0=-2$

$$\left|\frac{a_{n+1}}{a_n}\right| = \sqrt{\frac{n+1}{n}}\cdot\left|\frac{(-5)^{\cancel{n+3}}}{5^{\cancel{2n}+2}}\cdot\frac{\cancel{5^{2n}}}{\cancel{(-5)^{n+2}}}\right| = \underbrace{\sqrt{\frac{n+1}{n}}}_{\downarrow\, 1}\left|\frac{-5}{25}\right| \xrightarrow{n\to\infty} \frac{1}{5}$$

$\Rightarrow \underline{R=5}$ (3)

Végpontok: $x_1=x_0-R=-2-5=-7$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^{n+2}}{5^{2n}\sqrt{n}}\underbrace{(-7+2)^n}_{-5}=\sum_{n=1}^{\infty}\frac{25}{\sqrt{n}}=\infty$ (2)

$x_2=x_0+R=-2+5=+3$

$\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^{n+2}}{5^{2n}\sqrt{n}}\underbrace{(+3+2)^n}_{5}=\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 25}{\sqrt{n}}$ konvergens, mert Leibniz (2)

Tehát $\underline{\underline{K.T.=(-7,+3]}}$ (1)

$\boxed{2}$ A hatványsor a konvergenciatartományának a <u>belsejében</u> bármely zárt intervallumon egyenletesen konvergens, tehát pl. $[-6,+2]$-n egy. konv.

[-5-]

4, a, $x = r\cos\varphi;\ y = r\sin\varphi$

$\boxed{5}$ $f(x,y) = \dfrac{\cancel{r^2}(2\cos^2\varphi + 3\sin^2\varphi)}{\cancel{r^2}(4\cos^2\varphi + \sin^2\varphi)}$ függ φ-től, tehát $\nexists \lim\limits_{(x,y)\to(0,0)} f(x,y)$

b,

$\boxed{7}$ $\dfrac{df}{d\underline{\ell}} = \lim\limits_{t\to 0+} \dfrac{1}{t}\Big(f\big(\tfrac{t}{\sqrt5}, \tfrac{2t}{\sqrt5}\big) - \underset{f(0,0)}{c}\Big)$ 2 $= \lim\limits_{t\to 0+} \dfrac{1}{t}\left(\dfrac{\cancel{t^2}\cdot\left(\frac{2}{5} + 3\,\frac{4}{5}\right)}{\cancel{t^2}\left(\frac{4}{5} + \frac{4}{5}\right)} - c\right) =$

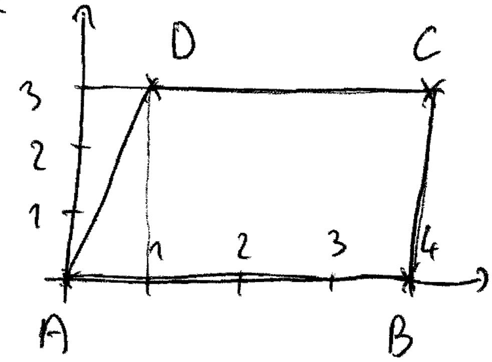
$= \lim\limits_{t\to 0+} \dfrac{1}{t}\cdot\left(\dfrac{14}{8} - c\right)$ 3 $= \lim\limits_{t\to 0+} \dfrac{7/4 - c}{t} = \begin{cases} 0, & \text{ha } c = 7/4 \\ \nexists & \text{egyébként} \end{cases}$

Csak akkor létezik a határérték, ha a számláló nulla, azaz $c = \dfrac{7}{4}$ esetén. Ekkor az iránymenti derivált 0. 2

5, $\boxed{5}$ $f(x) = e^{3x} = e^{3(x+4)}\cdot e^{-12} = e^{-12}\sum\limits_{n=0}^{\infty} \dfrac{3^n}{n!}(x+4)^n$ 4 ; $R = \infty$ 1

$\boxed{6}$ $g(x) = \sin x\cos x = \dfrac{1}{2}\sin(2x)$ 2 $= \dfrac{1}{2}\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n\cdot 2^{2n+1}}{(2n+1)!}x^{2n+1}$ 3 ; $R = \infty$ 1

6, *



$I = \iint\limits_T e^{6x-y}\,dT =$

$= \int\limits_{x=0}^{1}\left(\int\limits_{y=0}^{3x} e^{6x-y}\,dy\right)dx +$

$+ \int\limits_{x=1}^{4}\left(\int\limits_{y=0}^{3} e^{6x-y}\,dy\right)dx$ 3 $=$

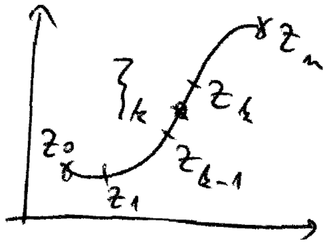
$= \int\limits_{y=0}^{3}\left(\int\limits_{x=\frac{y}{3}}^{4} e^{6x-y}\,dx\right)dy$ 2 $= \int\limits_{y=0}^{3} e^{-y}\underbrace{\left[\dfrac{e^{6x}}{6}\right]_{y/3}^{4}}_{\frac{e^{24}}{6} - \frac{e^{2y}}{6}}dy = \int\limits_{y=0}^{3}\left(\dfrac{e^{24}}{6}e^{-y} - \dfrac{1}{6}e^{y}\right)dy$ 3 $=$

6.º (folyt.)

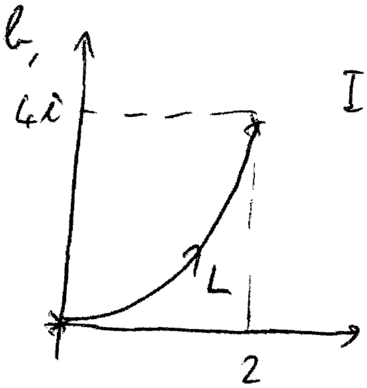
$$= -\frac{e^{24}}{6}\left[e^{-y}\right]_0^3 - \frac{1}{6}\left[e^{y}\right]_0^3 = -\frac{e^{24}}{6}(e^{-3}-1) - \frac{1}{6}(e^{3}-1) = \frac{+e^{24}-e^{21}-e^{3}}{6}$$ (3)

7* a, [4] $\int_L f(z)\,dz = \lim_{\max|\Delta z_k|\to 0} \sum_{k=1}^{n} (z_k - z_{k-1})\, f(\zeta_k)$ (3), ahol $\{z_k\}_{k=0}^{n}$ az L

egy felosztása, $\{\zeta_k\}_{k=1}^{n}$ reprezentáns pontok (1)



b, [10]



$$I = \int_L (\cos(2z) + \bar z)\,dz = \underbrace{\int_L \cos(2z)\,dz}_{I_1} + \underbrace{\int_L \bar z\,dz}_{I_2}$$

$$I_1 = \left[\frac{\sin(2z)}{2}\right]_0^{2+4i} = \frac{1}{2}\sin(4+8i) =$$ (3)

$$= \frac{1}{2}\sin 4\cos(8i) + \frac{1}{2}\cos 4\sin(8i) = \frac{\sin 4\,\mathrm{ch}\,8}{2} + i\,\frac{\cos 4\,\mathrm{sh}\,8}{2}$$ (2)

$$I_2 = \int_L \bar z\,dz = \int_{t=0}^{2} \underbrace{(t-it^2)(1+2it)}_{2t^3+it^2+t}\,dt = \left[\frac{2t^4}{4} + i\,\frac{t^3}{3} + \frac{t^2}{2}\right]_0^2 =$$ (3)

$x = t \in [0,2]$

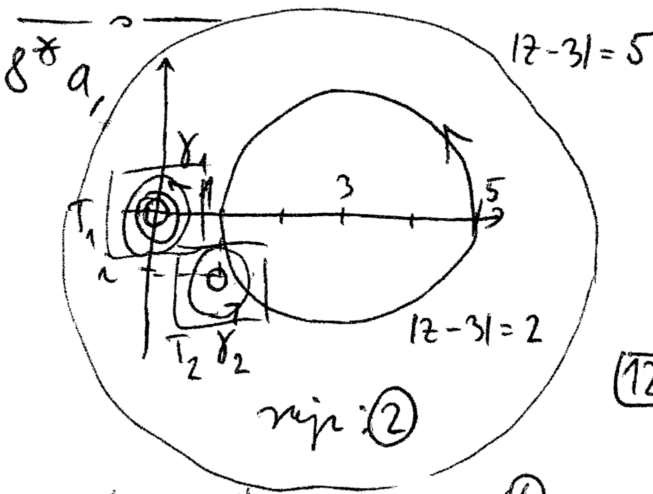
$$= 8 + \frac{8}{3}i + 2 = 10 + \frac{8}{3}i$$ (2)

$z(t) = t + it^2$; $\bar z = t - it^2$

$\dot z(t) = 1 + 2it$

$$I = I_1 + I_2$$

8* a,



a, $\oint f(z)\,dz = 0$, (1) mert f reguláris

$|z-3|=2$ a $|z-3|<2+\varepsilon$ tartományon.

(Cauchy-alaptétel)

b, [12] $$\oint_{|z-3|=5} f(z)\,dz = \oint_{\gamma_1} \frac{\frac{\sin(2z)}{z-1+i}}{z^2}\,dz + \oint_{\gamma_2} \frac{\left(\frac{\sin(2z)}{z^2}\right)}{z-(1-i)}\,dz$$ (4)

reg. T_1-en

$n=2$

$$= 2\pi i\left(\frac{\sin(2z)}{z-1+i}\right)'\Bigg|_0 + 2\pi i\,\frac{\sin(2z)}{z^2}\Bigg|_{1-i} = 2\pi i\left(\frac{2(\cancel{-1+i})}{(-1+i)^2} + \frac{\sin(2-2i)}{\underbrace{(1-i)^2}_{-2i}}\right) = 2\pi(-i+1) - \pi(\sin 2\,\mathrm{ch}\,2 - i\cos 2\,\mathrm{sh}\,2)$$ (4) (4)