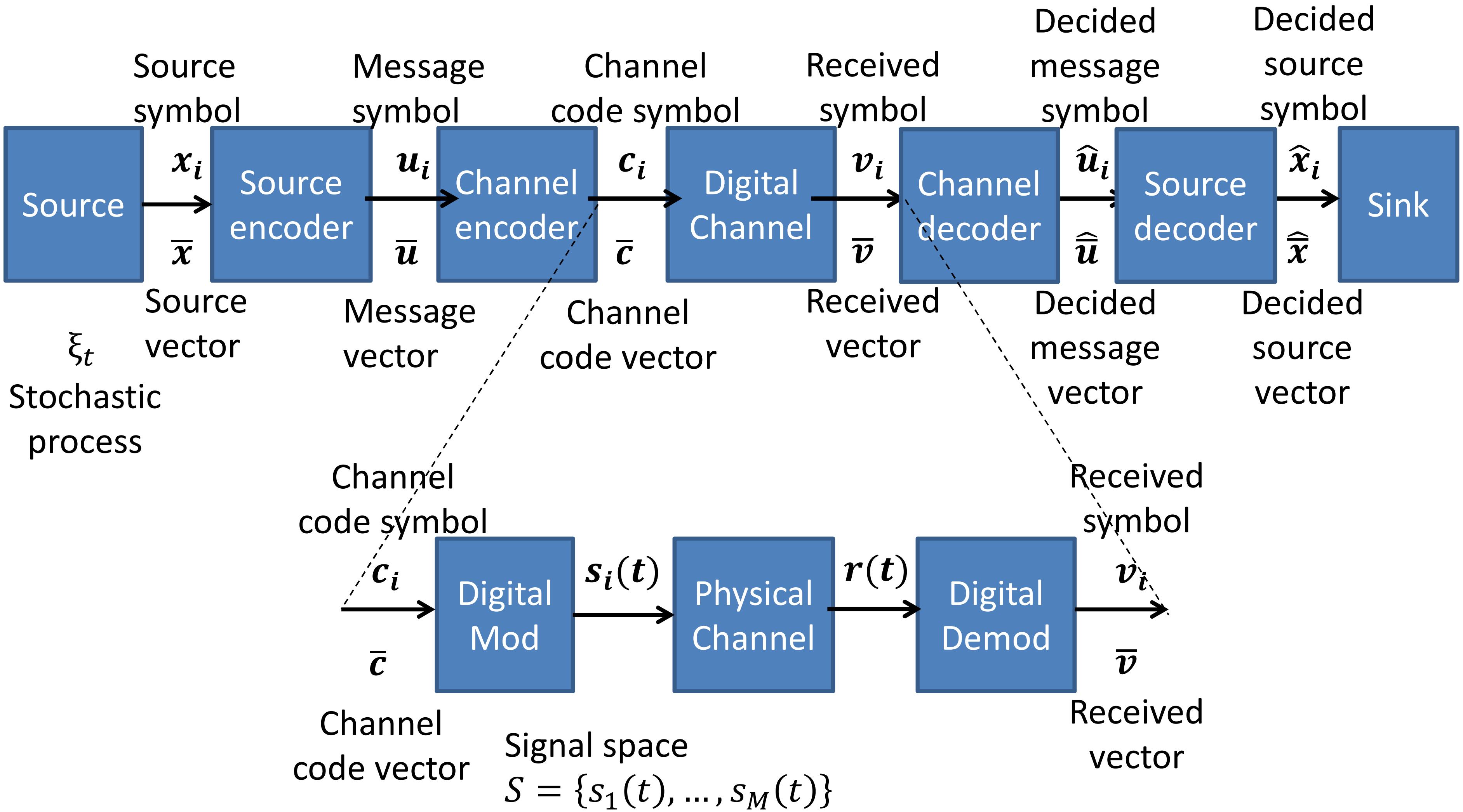
Űrkommunikáció
Space Communication
2023/10.

System overview
Modulation and Physical Channel



Capacity of the Physical Channel

- Until now: Investigation of time and value discrete channel
  - e.g. BSC, abstraction/model of real channels
- Now: Time and value continuous channel
  - because thermal noise is always present in physical systems

Entropy of continuous valued stochastic process (WSS, with $f_x(x)$ first order distribution)

$$H(\xi_x) \stackrel{\text{def}}{=} -E\{ld f_x(x)\} = \int_{x=-\infty}^{\infty} f_x(x) \cdot ld \frac{1}{f_x(x)} dx$$

Remark: Channel capacity = Amount of information can be transmitted

Shannon's Definition: Channel capacity $\left[\frac{Shannon}{channel\ use}\right], \left[\frac{bit}{channel\ use}\right]$, *e.g.* [bit/sec]

Discrete Channel,
X and Y are discrete RVs

$$C = \max_{p(x)} I(X\,;Y) =$$
$$= \max_{p(x)} [H(X) - H(X|Y)] =$$
$$= \max_{p(x)} [H(Y) - H(Y|X)] =$$
$$= \max_{p(x)} D\left(p\left(x_i, y_j\right) \,||\, p(x_i) \cdot p(y_j)\right)$$

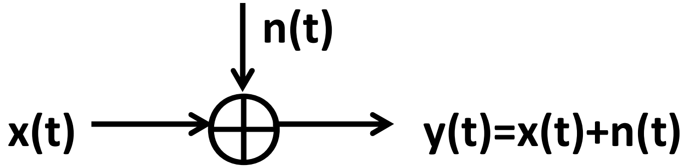
Continuous Channel
$x(t)$ and $y(t)$ are continuous stochastic processes with $f_x(x)$ and $f_y(y)$ first order distribution

$$C = \max_{f_x(x)} \tilde{I}\left(\xi_x; \xi_y\right) =$$
$$= \max_{f_x(x)} [H(\xi_x) - H(\xi_x|\xi_y)] =$$
$$= \max_{f_x(x)} [H\left(\xi_y\right) - H\left(\xi_y|\xi_x\right)] =$$
$$= \max_{f_x(x)} D\left(f_{x,y}(x,y) || f_x(x) \cdot f_y(y)\right)$$

AWGN Channel

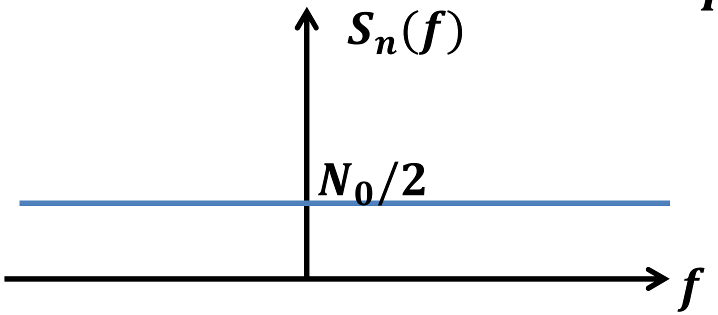
Additive White Gaussian Noise: AWGN

- Additive:

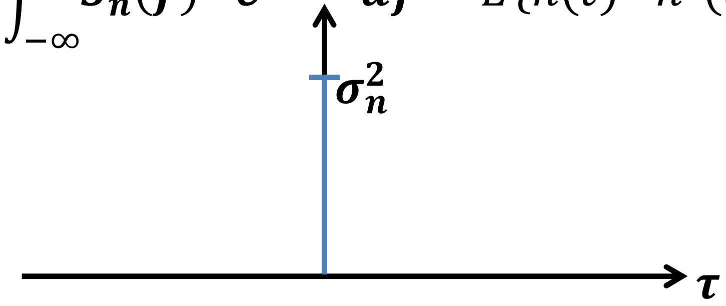


- White ⇔ constant Power Spectral Density, Auto-Correlation function Dirac:

$$\boldsymbol{R_{nn}(\tau)} = \int_{-\infty}^{\infty} \boldsymbol{S_n(f) \cdot e^{j2\pi f\tau} df} = E\{n(t) \cdot n^*(t-\tau)\}$$



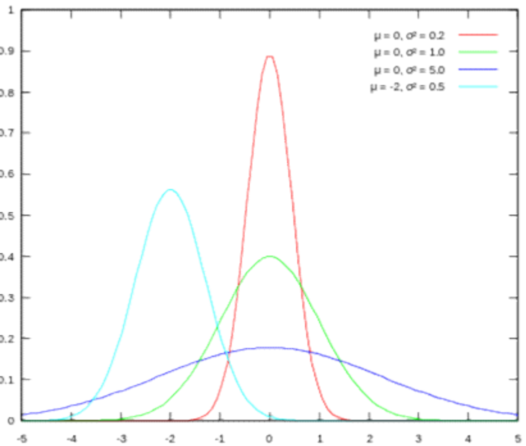
$\overset{\mathcal{F}}{\Leftrightarrow}$



$$N_0 = \frac{h \cdot f}{exp\left(\frac{h \cdot f}{k \cdot T_0}\right) - 1}$$

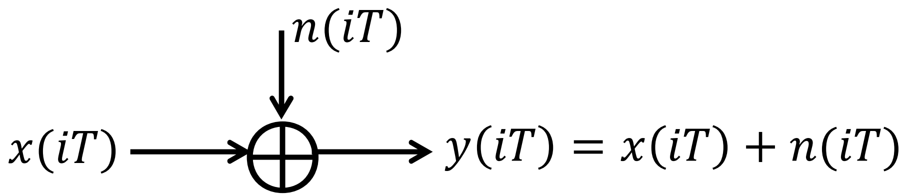
h – Planck constant
k – Boltzmann constant
T_0– ca. 19°C

- Gaussian: $G\left(\mu_n = 0, \sigma_n^2 = {}^{N_0}/_{2}\right)$



$$f_n^{(1)}(n) = \frac{1}{\sqrt{2\pi\sigma_n^2}} exp\left(-\frac{n^2}{2\sigma_n^2}\right)$$

Capacity of Time discrete AWGN, D-AWGN



$$C_{D-AW} = \max_{f_x(x)} [H(\xi_y) - H(\xi_y|\xi_x)] = \max_{f_x(x)} [H(\xi_y = \xi_x + \xi_n) - H(\xi_n)]$$

It can be proved, that C_{AWGN} is maximal, if the input Gaussian $G(\mu_x = 0, \sigma_x^2)$

$$f_x^{(1)}(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} exp\left(-\frac{x^2}{2\sigma_x^2}\right)$$

Let P_x the average power of the input (valid for ergodic process)

$$P_x = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} x_i^2 = \sigma_x^2 = \int_{-\infty}^{\infty} x^2 f_x^{(1)}(x)dx$$

The Entropy of the input:

$$H(\xi_x) = \int_{-\infty}^{\infty} f_x^{(1)}(x) \cdot ld \frac{1}{f_x^{(1)}(x)} dx$$

Capacity of Time discrete AWGN, D-AWGN

The Entropy of the input:

$$H(\xi_x) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} exp\left(-\frac{x^2}{2\sigma_x^2}\right) \cdot \underbrace{ld\left(\frac{1}{\sqrt{2\pi\sigma_x^2}} exp\left(-\frac{x^2}{2\sigma_x^2}\right)\right)}_{\frac{1}{\ln 2}\left[ln\left(\frac{1}{\sqrt{2\pi\sigma_x^2}}\right) - \frac{x^2}{2\sigma_x^2}\right]} dx =$$

$$= -\frac{1}{\ln 2} ln\left(\frac{1}{\sqrt{2\pi\sigma_x^2}}\right) \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_x^2}} exp\left(-\frac{x^2}{2\sigma_x^2}\right) dx}_{1} + \frac{1}{ln2} \cdot \int_{-\infty}^{\infty} \frac{x^2}{2\sqrt{2\pi}\sigma_x^3} exp\left(-\frac{x^2}{2\sigma_x^2}\right) dx =$$

$$= ld\sqrt{2\pi\sigma_x^2} + \frac{2}{2 \cdot ln2} \cdot \frac{1}{\sqrt{2\pi}\sigma_x^3} \cdot \underbrace{\int_{0}^{\infty} x^2 \cdot exp\left(-\frac{x^2}{2\sigma_x^2}\right) dx}_{\Gamma\ function\ at\ m=2, a=\frac{1}{2\cdot\sigma_x^2}} =$$

$$= \frac{1}{2} \cdot ld(2\pi\sigma_x^2) + \frac{1}{ln2} \cdot \frac{1}{\sqrt{2\pi}\sigma_x^3} \cdot \frac{\sqrt{\pi}}{4 \cdot a^{3/2}} = \frac{1}{2} \cdot ld(2\pi\sigma_x^2) + \frac{1}{ln2} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_x^3} \cdot \frac{(2 \cdot \sigma_x^2)^{3/2}\sqrt{\pi}}{4}}_{1/2}$$

$$= \frac{1}{2} ld(2\pi\sigma_x^2) + \frac{1}{2} \cdot \frac{\overbrace{\ln e}^{ld\ e}}{ln2} = \frac{1}{2} ld(2\pi e\sigma_x^2)$$

Capacity of Time discrete AWGN, D-AWGN

The Entropy of the input:

$$H(\xi_x) = \frac{1}{2} ld(2\pi e \sigma_x^2)$$

Because $y(iT) = x(iT) + n(iT)$ and ξ_x and ξ_n are independent with

$$G(\mu_x = 0, \sigma_x^2 = P_x) \text{ and } G\left(\mu_n = 0, \sigma_n^2 = {}^{N_0}/_2\right)$$

respectively, therefore the output ξ_y follows also $G\left(\mu_y = 0, \sigma_y^2 = \sigma_x^2 + \sigma_n^2\right)$

Capacity of Time discrete AWGN, D-AWGN:

$$C_{D-AWG} = \max_{f_x(x)} [H(\xi_y) - H(\xi_y|\xi_x)] = \max_{f_x(x)} [H(\xi_y = \xi_x + \xi_n) - H(\xi_n)]$$

Recap: C_{AWGN} is maximal, if the input Gaussian

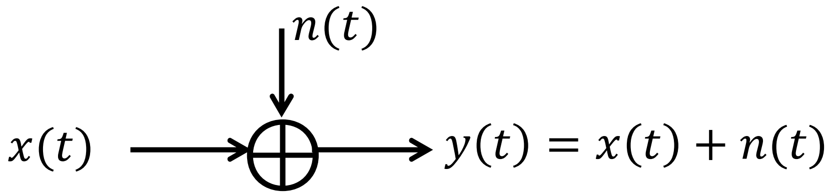
$$C_{D-AWG} = H(\xi_y = \xi_x + \xi_n) - H(\xi_n) = \frac{1}{2} ld(2\pi e[P_x + \sigma_n^2]) - \frac{1}{2} ld(2\pi e \sigma_n^2) =$$

$$C_{D-AW} = \frac{1}{2} ld \frac{2\pi e[P_x + \sigma_n^2]}{2\pi e \sigma_n^2} = \frac{1}{2} ld\left(1 + \frac{P_x}{\sigma_n^2}\right) \left[\frac{bit}{channel\ use}\right]$$

Recap: average input power P_x [Watt], noise power spectral density $\sigma_n^2 = N_0/2$ [Watt/Hz]

=> P_x/σ_n^2 [bit/sec]

Capacity of Time continuous, Bandlimited AWGN



$$C_{AWGN} = \lim_{T\to\infty} \max_{f_x(x)} \frac{1}{T} \tilde{I}(\xi_x; \xi_y)$$

- Duration of communication: T
- Spectral Bandwidth of ξ_x: B
- Recap: Shannon's sampling theorem: $T_s \leq 1/2B$

- Representing time continuous signals with their samples:
  - $x(t) \leftrightarrow \bar{X} = [X_1, X_2, \dots, X_K]$
  - $y(t) \leftrightarrow \bar{Y} = [Y_1, Y_2, \dots, Y_K]$
  - $n(t) \leftrightarrow \bar{N} = [N_1, N_2, \dots, N_K]$

Where (by sampling at the Nyquist rate): $K \cdot T_s = T \Rightarrow K = T/T_s = 2BT$

Limit of average input power:

$$P_{av} = \lim_{T\to\infty} \frac{1}{T} \int_0^T E\{x^2(t)\}\, dt = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{K} E\{x_i^2\} \overset{WS}{\Longleftrightarrow} \lim_{T\to\infty} \frac{K \cdot \sigma_x^2}{T} = \lim_{T\to\infty} \frac{2BT\sigma_x^2}{T} = 2B\sigma_x^2$$

Capacity of Time continuous, Bandlimited AWGN

Limit of average input power:

$$P_{av} = 2B\sigma_x^2 \Longrightarrow \sigma_x^2 = P_{av}/2B$$

Capacity:

$$C_{AWGN} = \lim_{T\to\infty} \frac{1}{T} \max_{f_x(x)} \tilde{I}\left(\xi_x; \xi_y\right) = \lim_{T\to\infty} \frac{1}{T} \max_{f_x(x)} \left[H\left(\xi_y = \xi_x + \xi_n\right) - H(\xi_n)\right] =$$

$$= \lim_{T\to\infty} \frac{1}{T} \max_{f_x(x)} \sum_{i=1}^{K} \left[H(Y_i = X_i + N_i) - H(N_i)\right] = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{K} \underbrace{\max_{f_x(x)} \left[H(Y_i = X_i + N_i) - H(N_i)\right]}_{C_{D-AWGN}} =$$

$$= \lim_{T\to\infty} \frac{1}{T} \cdot K \cdot \frac{1}{2} ld\left(1 + \frac{\sigma_x^2}{\sigma_n^2}\right) = \lim_{T\to\infty} \frac{2BT}{2T} \cdot ld\left(1 + \frac{P_{av}}{2B\sigma_n^2}\right) = B \cdot ld\left(1 + \frac{P_{av}}{2B\sigma_n^2}\right)$$

Recap: Power Spectral Density constant $\sigma_n^2 = N_0/2$

- average input power: P_{av}
- noise power: $2B\sigma_n^2 = B \cdot N_0$

Signal to Noise Ratio, SNR$= \dfrac{P_{av}}{BN_0}$

Capacity of time continuous, Bandlimited AWGN with power constrain P_{av}

$$C_{AWGN} = B \cdot ld\left(1 + \frac{P_{av}}{BN_0}\right) \quad \left[\frac{bit}{sec}\right]$$

Capacity of bandlimited, power constrained AWGN

Capacity limits:

- $C_{AWGN} \uparrow \quad when \ \ P_{av} \uparrow \qquad\qquad and\ B\ constant$

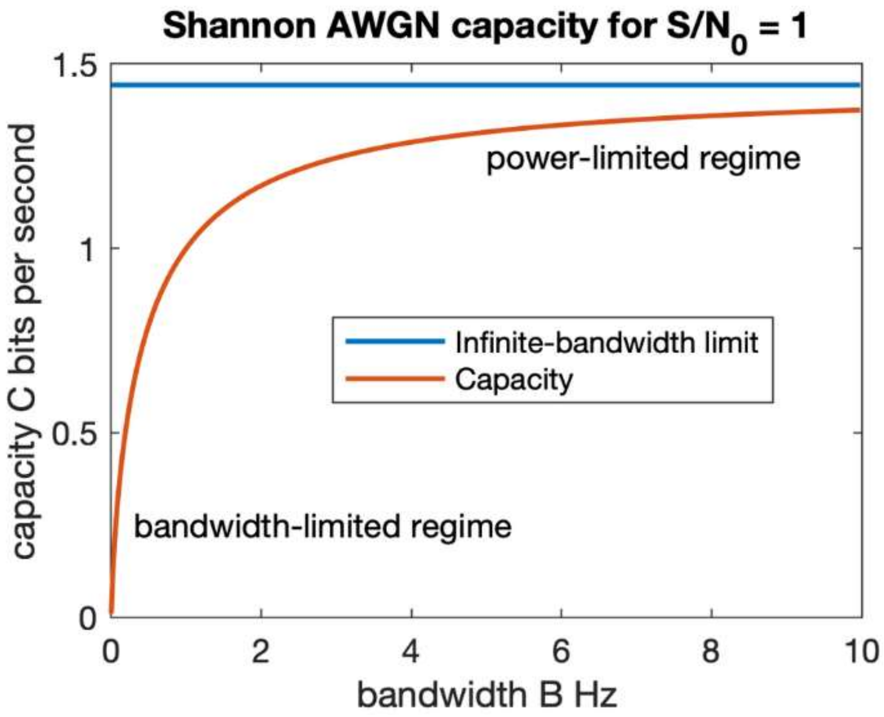
or

- $C_{AWGN} \uparrow \quad when \ \ P_{av}\ constant, and\ B \uparrow$

Or both

$$C_\infty = \lim_{B\to\infty} C_{AWGN} = \lim_{B\to\infty} B \cdot ld\left(1 + \frac{P_{av}}{BN_0}\right) = \frac{P_{av}}{N_0} \cdot ld\ e = \frac{P_{av}}{N_0 \cdot ln2} \quad \left[\frac{bit}{sec}\right]$$

1/ln2=1,4426950408…



Shannon's Channel Capacity Limit

Bandlimited, Power Constrained AWGN

- Normalized channel capacity: C/B [bit/sec/Hz], [Shannon/sec/Hz]
- Bit energy: E_b [Joule]
- Constrained average input power: $P_{av} = E_b \cdot C$ [Watt] [Joule/sec]

- Signal to Noise Ratio, SNR$= \frac{P_{av}}{BN_0} = \frac{C \cdot E_b}{B \cdot N_0} \Rightarrow$ bit-SNR$= \frac{E_b}{N_0}$, SNR/bit/sec/Hz

Recap: $C_{AWGN} = B \cdot ld\left(1 + \frac{P_{av}}{BN_0}\right) \quad \left[\frac{bit}{sec}\right]$

$$C/B = ld\left(1 + \frac{P_{av}}{BN_0}\right) \overset{2^\wedge}{\Rightarrow} 2^{C/B} = 1 + \frac{E_b \cdot C}{B \cdot N_0} \Rightarrow \frac{E_b}{N_0} = \frac{2^{C/B} - 1}{C/B}$$

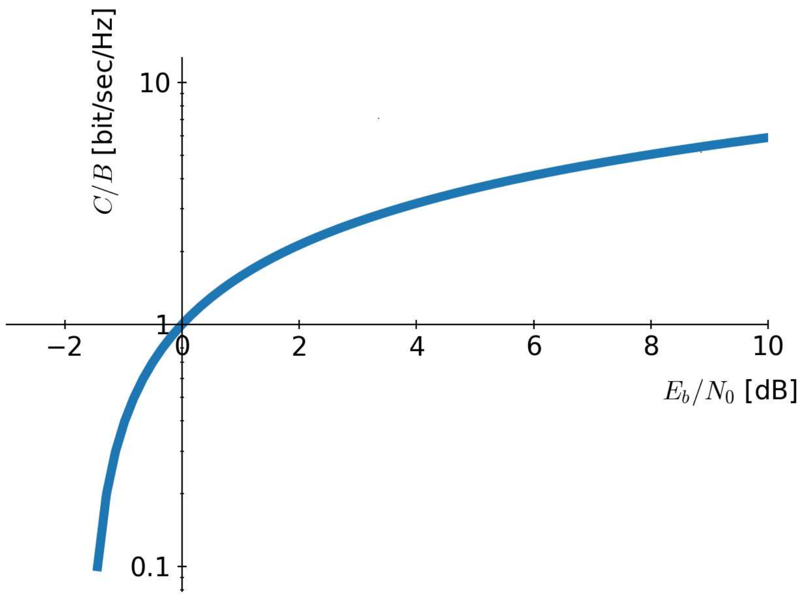
- $C/B = 1 \quad \Rightarrow \quad E_b/N_0 = 1 \; (0 \; dB)$

- $C/B \to \infty \quad \Rightarrow \quad E_b/N_0 \approx exp\left[\frac{C}{B} \cdot ln2 - \ln \frac{C}{B}\right]$

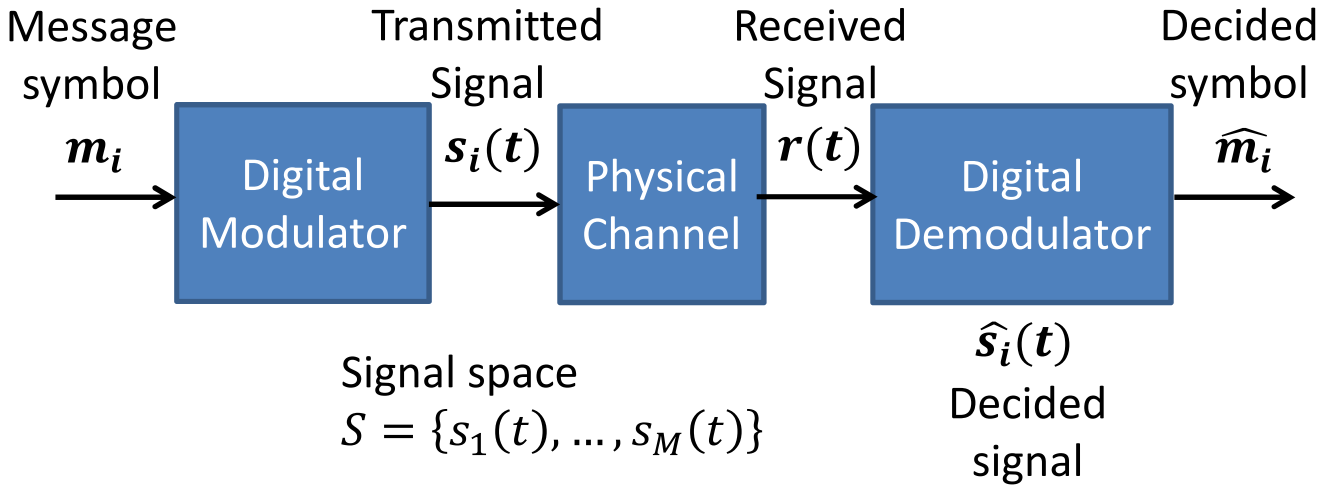
  increasing exponentially

- $C/B \to 0 \quad \Rightarrow \quad E_b/N_0 = \lim_{C/B \to \emptyset} \frac{2^{C/B} - 1}{C/B} = \ln 2$

  (≅-1.6 dB)



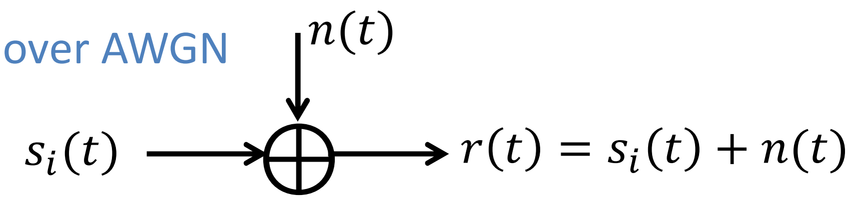
Digital Modulation



M-ary Digital Modulation

- Set of possible messages: $\mathcal{M} = \{m_1, \dots, m_M\}$
- Message symbol duration: T_s
- a-priori probability distribution of messages: $\mathrm{P} = \{p_1 = p(m_1), \dots, p_M = p(m_M)\}$
- Set of Digital Signals: $S = \{s_1(t), \dots, s_M(t)\}$

M-ary Digital Modulation over AWGN



Digital Modulation

Definition of

M-ary Digital Signal set: $S = \{s_1(t), \dots, s_M(t)\}$

- Finite set of different signals, M different waveforms:
$$s_i(t) \neq s_j(t) \quad \forall\, i,j \quad i \neq j$$
- Finite signal duration T_s:
$$s_i(t) \equiv 0 \quad \forall\, t < 0 \; and \; t \geq T_s \quad \forall i$$
- Finite signal energy E_i:
$$E_i = \int_{t=0}^{T_s} s_i^2(t)dt \leq E_{max} < \infty \quad \forall\, i$$

Modulation dimension, Dimension of M-ary Digital Signal set *D:*

- If the signals of the set are orthogonal, then D=M
$$\int_{t=0}^{T_s} s_i(t) \cdot s_j(t)\, dt = \begin{cases} 0 & \forall\, i \neq j \\ E_i & i = j \end{cases}$$
- Else: non-orthogonal signal set: D<M
  - Gram–Schmidt process for orthonormalising a set

Digital Modulation

Gram–Schmidt process for orthonormalising a set

- Project an M-ary signal set to a D<M dimensional signal space

$$S = \{s_1(t), \dots, s_M(t)\} \to \Phi = \{\varphi_1(t), \dots, \varphi_D(t)\}$$

- Basis functions of a D dimensional signal space Φ
  - Orthogonal and normalized functions

$$\int_{t=0}^{T_s} \varphi_i(t) \cdot \varphi_j(t)\, dt = \begin{cases} 0 & \forall\, i \neq j \\ 1 & i = j \end{cases}$$

Steps of Gram–Schmidt process

a) Normalization of the first digital signal of the set S

b) Calculate the projection of the second (next) digital signal into $\varphi_1(t)$ (already defined part of Φ)

c) Define the second (next) basis function as the normalized orthogonal component of the second (next) digital signal

d) Repeat b)-c) until the projection of all digital signal of the set are processed

Gram–Schmidt process for orthonormalising a set

a)

$$\varphi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \rightarrow \int_{t=0}^{T_s} \varphi_1^2(t)dt = 1 \rightarrow s_1(t) = \sqrt{E_1} \cdot \varphi_1(t) = s_{11} \cdot \varphi_1(t)$$

b)

$$s_{21} = \int_{t=0}^{T_s} s_2(t) \cdot \varphi_1(t)\, dt$$

c)

$$\acute{\varphi}_2(t) = s_2(t) - s_{21} \cdot \varphi_1(t) \rightarrow \varphi_2(t) = \frac{\acute{\varphi}_2(t)}{\underbrace{\sqrt{\int_{t=0}^{T_s} \acute{\varphi}_2^2(t)dt}}_{s_{22}}} \rightarrow s_2(t) = s_{21} \cdot \varphi_1(t) + s_{22} \cdot \varphi_2(t)$$

d) In general $s_i(t) \rightarrow \Phi$

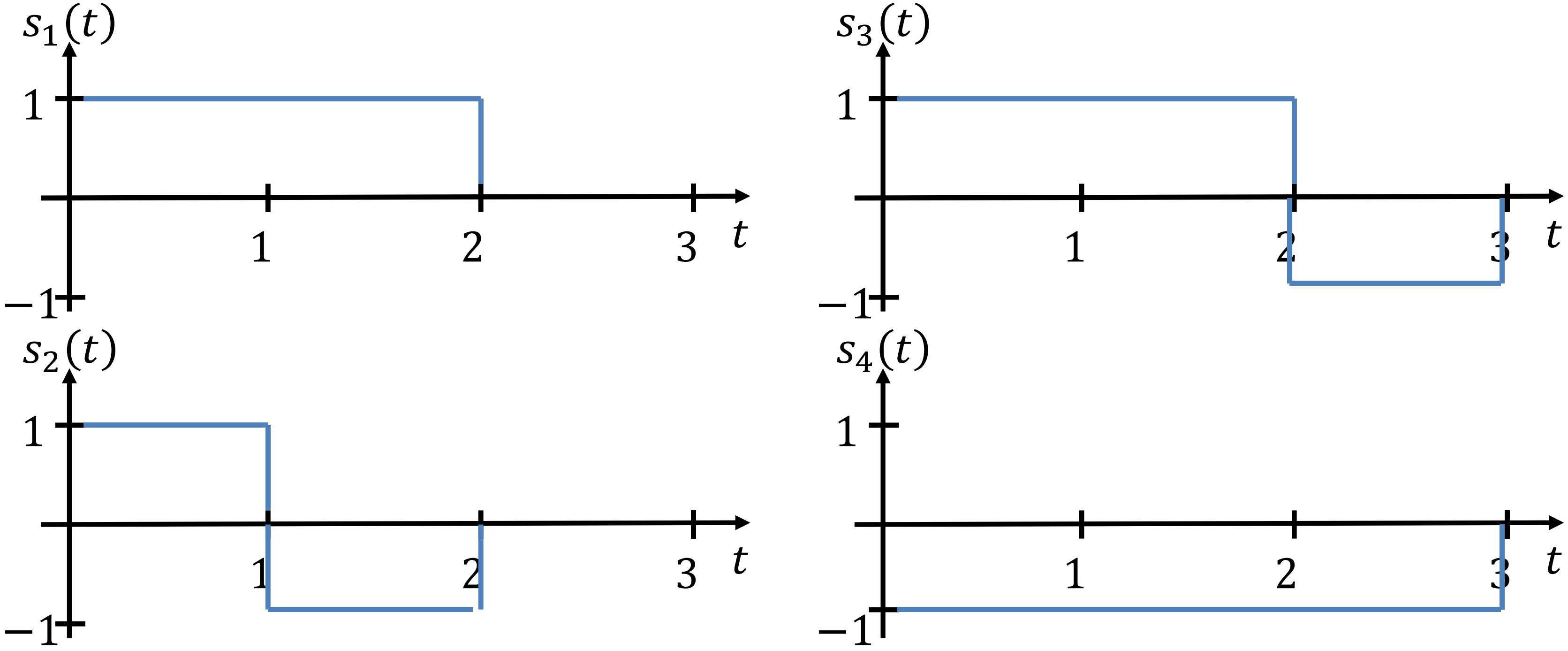
$$s_{ik} = \int_{t=0}^{T_s} s_i(t) \cdot \varphi_k(t)\, dt$$

$$\acute{\varphi}_j(t) = s_i(t) - \sum_{k=1}^{j-1} s_{ik} \cdot \varphi_k(t) \rightarrow \varphi_j(t) = \frac{\acute{\varphi}_j(t)}{\underbrace{\sqrt{\int_{t=0}^{T_s} \acute{\varphi}_j^2(t)dt}}_{s_{ij}}}$$

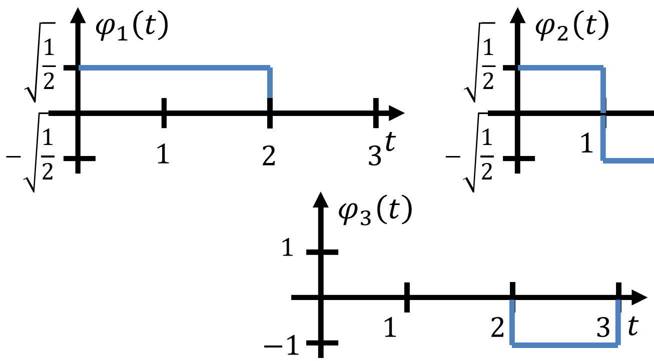
Digital Modulation

Example: Gram–Schmidt process for orthonormalising a set

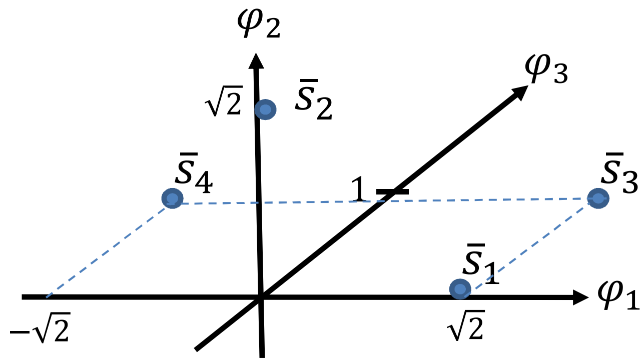
Quarternary digital signal set



Basis functions of a 3-dimensional signal space Φ



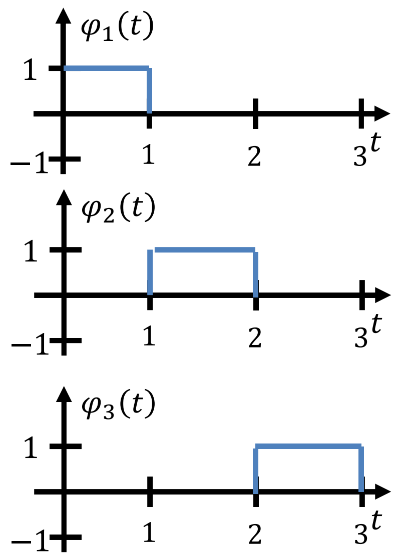
4 signals in the signal space Φ



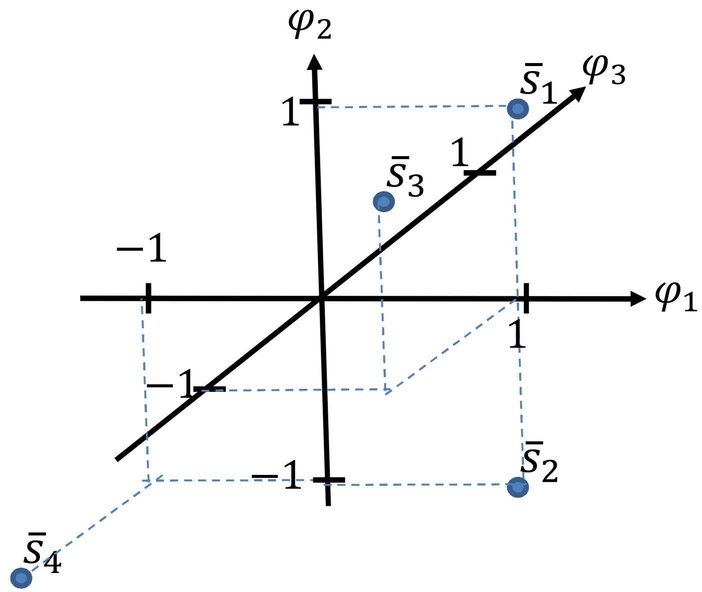
Digital Modulation

Example: Heuristic orthonormal basic functions

Basis functions of a 3-dimensional signal space Φ



4 signals in the signal space Φ

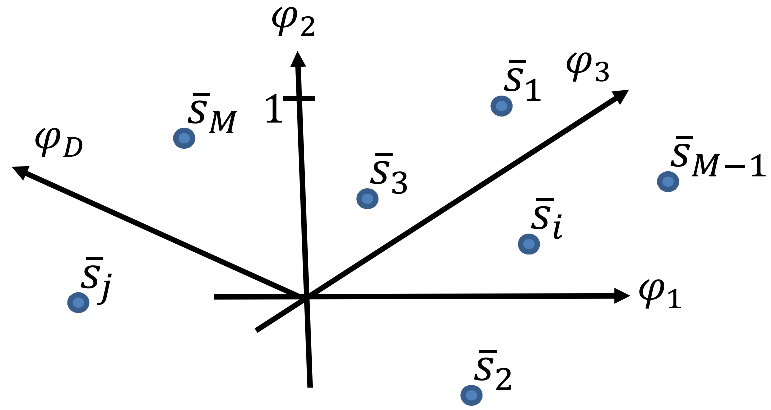


Representation of a D dimensional, M-ary digital signal set in Signal Vector Space Φ

$$S = \{s_1(t), \dots, s_M(t)\} \leftrightarrow \bar{S} = \{\bar{s}_1, \dots, \bar{s}_M\}$$

$$s_i(t) = \sum_{k=1}^{D} s_{ik} \cdot \varphi_k(t)$$

$$s_i(t) \leftrightarrow \bar{s}_i = [s_{i1}, \dots, s_{iD}]$$



D dimensional, M-ary Signal Vectors

- Signal energy $E_i \equiv |\bar{s}_i|^2$:

$$E_i = \int_{t=0}^{T_s} s_i^2(t)dt = \int_{t=0}^{T_s} \left[\sum_{k=1}^{D} s_{ik} \cdot \varphi_k(t)\right]^2 dt \overset{orthgonal}{\triangleq} \int_{t=0}^{T_s} \sum_{k=1}^{D} s_{ik}^2 \cdot \varphi_k^2(t)\, dt =$$

$$= \sum_{k=1}^{D} s_{ik}^2 \cdot \int_{t=0}^{T_s} \varphi_k^2(t)dt \overset{normalised}{\triangleq} \sum_{k=1}^{D} s_{ik}^2 = |\bar{s}_i|^2$$

Where $|\bar{s}_i|^2$ is the square of the absolute value (length) of the signal vector $\bar{s}_i$

- Correlation of time synchronic digital signals $R_{s_i s_j}(\tau = 0) \equiv \bar{s}_i \cdot \bar{s}_j$:

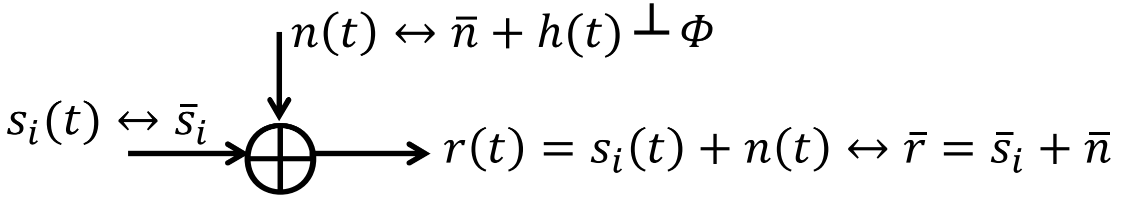
$$R_{s_i s_j}(\tau = 0) = \int_{t=0}^{T_s} s_i(t) \cdot s_j(t)dt = \int_{t=0}^{T_s} \left(\sum_{k=1}^{D} s_{ik} \cdot \varphi_k(t)\right) \cdot \left(\sum_{k=1}^{D} s_{jk} \cdot \varphi_k(t)\right) dt =$$

$$= \sum_{k=1}^{D} s_{ik} \cdot s_{jk} = \bar{s}_i \cdot \bar{s}_j$$

Where $\bar{s}_i \cdot \bar{s}_j$ is the scalar product of the vectors

Vectors in signal space Φ

Signals, noise and received signal in case of AWGN



- *Signal*

$S = \{s_1(t), \ldots, s_M(t)\} \leftrightarrow \bar{S} = \{\bar{s}_1, \ldots, \bar{s}_M\}$

$s_i(t) \leftrightarrow \bar{s}_i = [s_{i1}, \ldots, s_{iD}]$

- *Noise*

$$n_k = \int_{t=0}^{T_s} n(t) \cdot \varphi_k(t)\, dt$$

$$n(t) = \sum_{k=1}^{D} n_k \cdot \varphi_k(t) + h(t) \perp \Phi$$

$n(t) \leftrightarrow \bar{n} = [n_1, \ldots, n_D]$

- *Received Signal*

$$r_k = \int_{t=0}^{T_s} r(t) \cdot \varphi_k(t)\, dt$$

$r(t) \leftrightarrow \bar{r} = [r_1, \ldots, r_D]$