Infó an. vizs. MINTAMEGOLDÁS | C-1-1 | 2015. 06. 18.

1, a, $\boxed{2}$ Ha $\sum_{n=0}^{\infty} a_n$ konvergens, akkor $\lim_{n\to\infty} a_n = 0$.

b, $\boxed{6}$

$$S_{n+1} = S_n + a_{n+1} \quad / \lim_{n\to\infty}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$S = S + \lim_{n\to\infty} a_{n+1} \implies \lim_{n\to\infty} a_n = 0$$

c, $\boxed{3}$ $\sum_{n=1}^{\infty} \underbrace{(-1)^n \frac{1}{\sqrt[n]{9n}}}_{a_n}$ divergens, mivel $\lim_{n\to\infty} \frac{1}{\sqrt[n]{9n}} = 1$, tehát $\lim_{n\to\infty} a_n \neq 0$.

2, a, $\boxed{5}$

$$\lim_{n\to\infty} \left(\frac{n+4}{n+3}\right)^{2n} = \lim_{n\to\infty} \left(\frac{\left(1+\frac{4}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n}\right)^2 \text{③} = \left(\frac{e^4}{e^3}\right)^2 = \underline{\underline{e^2}} \text{②}$$

b, $\boxed{5}$

$$\lim_{x\to 0+} x \ln(4x) = \lim_{x\to 0+} \frac{\ln(4x)}{(1/x)} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \text{②} \lim_{x\to 0+} \frac{(1/x)}{(-1/x^2)} \text{②} = \lim_{x\to 0+} -x = \underline{\underline{0}} \text{①}$$

3, a, $\boxed{8}$

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{(x+1)(x-1)} = 1 + \frac{A}{x+1} + \frac{B}{x-1} \text{③}$$

$$A(x-1) + B(x+1) = \underbrace{(A+B)}_{0} x + \underbrace{(B-A)}_{1} \overset{!}{=} 1 \implies \underline{A = -\frac{1}{2},\ B = \frac{1}{2}} \text{②}$$

$$\int_2^3 \frac{x^2}{x^2-1}\,dx = \int_2^3 1\,dx + \int_2^3 \left(\frac{-1/2}{x+1} + \frac{1/2}{x-1}\right) dx = 1 + \frac{1}{2}\Big[\ln|x+1| - \ln|x-1|\Big] \text{②}$$

$$= 1 + \frac{1}{2}\big(\ln 4 - \ln 3 - (\ln 2 - \ln 1)\big) \text{①} = 1 + \frac{\ln 4 - \ln 3 - \ln 2}{2} = \underline{\underline{1 + \ln\sqrt{\frac{2}{3}}}}$$

$\boxed{3}$ b,

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \underline{\underline{\frac{1}{2}\ln(x^2+1) + C}}$$

$\boxed{5}$ c,

$$\int \ln x\,dx = \int \underset{u'}{1} \cdot \underset{v}{\ln x}\,dx \text{②} = x \ln x - \int x \frac{1}{x}\,dx \text{②} = \underline{\underline{x \ln x - x + C}} \text{①}$$

$u = x \quad v' = \frac{1}{x}$

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4, [9]

$$y' = \sqrt{\underbrace{y^2+6xy+9x^2}_{(y+3x)^2}+1} - 3$$

$u(x) = y(x) + 3x$

$u' = y' + 3 \Rightarrow y' = u' - 3$

$$u' - \cancel{3} = \sqrt{u^2+1} - \cancel{3} \Rightarrow \int \frac{du}{\sqrt{u^2+1}} = \int dx \Rightarrow \text{arsh}\, u = x + C \quad (3)$$

(4)

$u(x) = \text{sh}(x+C)$; $\underline{\underline{y(x) = \text{sh}(x+C) - 3x}}$ (2) $(C \in \mathbb{R})$

— o —

5, a, [3] Az $S(x) = \sum_{n=0}^{\infty} f_n(x)$ függvénysor egyenletesen konvergens az $I \subset \mathbb{R}$ intervallumon, ha $\forall \varepsilon > 0$ esetén $\exists N(\varepsilon)$, hogy $\forall x \in I$, $\forall n > N(\varepsilon)$ esetén $|S_n(x) - S(x)| = \left|\sum_{k=0}^{n} f_k(x) - S(x)\right| < \varepsilon$. ($N(\varepsilon)$ független x-től.)

[3] b, T.: (Weierstrass) Ha $\forall n \in \mathbb{N}$ és $\forall x \in I$ esetén $|f_n(x)| \le b_n$, és $\sum_{n=0}^{\infty} b_n < \infty$, akkor $\sum_{n=0}^{\infty} f_n(x)$ abszolút és egyenletesen konv I-n.

[6] c, Ha $x \in [-\frac{1}{2}, +\frac{1}{2}]$, akkor $|f_n(x)| = |x^n| \le \frac{1}{2^n} =: b_n$,

és $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2 < \infty$, tehát a Weierstrass-tétel értelmében $\sum_{n=0}^{\infty} x^n$ egyenletesen konvergens $[-\frac{1}{2}, +\frac{1}{2}]$-n.

— o —

6, a, [4]

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \underline{\sum_{n=0}^{\infty} (-1)^n x^n}, \quad (3)$$

ha $|-x| < 1$, $\underline{\underline{R = 1}}$ (1)

[6] b,

$$g(x) = \ln(1+x) = g(0) + \int_0^x g'(t)\,dt = 0 + \int_0^x f(t)\,dt =$$

$$= \int_0^x \left(\sum_{n=0}^{\infty} (-1)^n t^n\right) dt \ (3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \ (2) = \underline{\underline{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}}}; \quad \frac{R = 1}{\text{(mint a, -ban)}} \ (1)$$

ha $|x| < 1$

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7*/ $\boxed{13}$

$$g(x,y)=(2x-y)^2+4x^3-6y$$

$$g'_x(x,y)=2\cdot 2(2x-y)+12x^2=8x+12x^2-4y=0 \quad (2)$$

$$g'_y(x,y)=-2(2x-y)-6=-4x+2y-6=0 \quad (2) \Longrightarrow y=2x+3$$

$$\cancel{8x}+12x^2-4(\cancel{2x}+3)=0 \Longrightarrow x^2=1 \Longrightarrow x_1=+1,\ y_1=5;\quad x_2=-1;\ y_2=1 \quad (3)$$

$$|\underline{\underline{H}}(x,y)|=\begin{vmatrix} g''_{xx} & g''_{xy} \\ g''_{yx} & g''_{yy}\end{vmatrix}=\begin{vmatrix} 8+24x & -4 \\ -4 & 2\end{vmatrix}=16+48x-16=48x \quad (2)$$

(1,5)-ben $\underline{\underline{H}}(1,5)=48>0 \Longrightarrow \exists$ lok. szél. érték

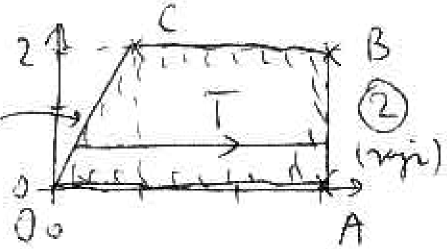
$g''(1,5)=8+24=32>0 \Longrightarrow$ lok. minimum van (2)

(−1,1)-ben $\underline{\underline{H}}(-1,1)=-48<0 \Longrightarrow$ nincs lok. szél. ért. (nyeregpont) (2)

8*/ $\boxed{11}$

$y=2x$

$x=\frac{y}{2}$



$$\iint_T e^{+5y-2x}\,dT=\int_{y=0}^{2}\left(\int_{x=y/2}^{3} e^{5y}e^{-2x}\,dx\right)dy= \quad (3)$$

$$=\int_{y=0}^{2} e^{5y}\left[\frac{e^{-2x}}{-2}\right]_{x=y/2}^{3}dy=\int_{y=0}^{2} e^{5y}\left(-\frac{1}{2}e^{-6}+\frac{1}{2}e^{-y}\right)dy= \quad (3)$$

$$=-\frac{e^{-6}}{2}\int_{y=0}^{2} e^{5y}\,dy+\frac{1}{2}\int_{y=0}^{2} e^{4y}\,dy=-\frac{e^{-6}}{2}\left[\frac{e^{5y}}{5}\right]_0^2+\frac{1}{2}\left[\frac{e^{4y}}{4}\right]_0^2=$$

$$=\frac{-e^{-6}}{10}\left(e^{10}-1\right)+\frac{1}{8}\left(e^{8}-1\right) \ (3) =\underline{\underline{\frac{-1}{10}e^{4}+\frac{1}{10}e^{-6}+\frac{1}{8}e^{8}-\frac{1}{8}}}$$

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9* a) [2]

$$\mathcal{F}[f'](\omega) = i\omega\, \mathcal{F}[f](\omega)$$

[6] b) $\mathcal{F}[f'](\omega) = \int\limits_{x=-\infty}^{\infty} e^{-i\omega x} f'(x)\, dx =$ parciális int.

$$= \lim_{\substack{R_1 \to -\infty \\ R_2 \to \infty}} \left(\left[e^{-i\omega x} f(x) \right]_{x=R_1}^{R_2} - \int\limits_{x=R_1}^{R_2} (-i\omega) e^{-i\omega x} f(x)\, dx \right) = \quad ③$$

$$= 0 + i\omega \int\limits_{-\infty}^{\infty} e^{-i\omega x} f(x)\, dx = i\omega\, \mathcal{F}[f](\omega) \quad ③$$

(A kiintegrált rész 0, mivel $|e^{i\omega x}| = 1$, és ha f Fourier-transzformálható, akkor $\lim\limits_{x \to \pm\infty} f(x) = 0$.)