INFO ANALÍZIS SZIGORLAT | -1- | (2017. 06. 07.

α VARIÁNS (Részletes)

1, $\boxed{13}$

a, T: (Rendőr elv): Ha $\forall n \in \mathbb{N}$-re (vagy $\forall n > N_0$-ra)

$a_n \le b_n \le c_n$, és $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = A$, akkor

$\exists \lim_{n\to\infty} b_n = A$ ③

⑤ B: Adott $\varepsilon > 0$. $\Rightarrow \exists N_a(\varepsilon)$: $\forall n > N_a(\varepsilon)$: $A-\varepsilon < a_n < A+\varepsilon$

$\exists N_c(\varepsilon)$: $\forall n > N_c(\varepsilon)$: $A-\varepsilon < c_n < A+\varepsilon$

Legyen $N(\varepsilon) = \max\{N_a(\varepsilon), N_c(\varepsilon), N_0\}$. Ekkor $\forall n > N(\varepsilon)$:

$A-\varepsilon < a_n < b_n < c_n < A+\varepsilon \Rightarrow \exists \lim_{n\to\infty} b_n = A$.

b,

$$\frac{9}{2} = \sqrt[n]{\frac{3\cdot 9^n - 9^n}{2^n + 2^n}} \underset{\text{Ha } n > N_0}{\le} \underbrace{\sqrt[n]{\frac{3^{2n+1} - 3n^2 + 2}{2^n + 6}}}_{a_n} = \sqrt[n]{\frac{3\cdot 9^n - 3n^2 + 2}{2^n + 6}} \le \frac{\sqrt[n]{3}\cdot 9}{2} \to \frac{9}{2}$$ ⑤

Tehát $\lim_n a_n = \underline{\underline{\frac{9}{2}}}$

2, a, D: $\lim_{x\to x_0} f(x) = A$, ha

③ i, x_0 a D_f torlódási pontja és

ii, $\forall \varepsilon > 0$: $\exists \delta(\varepsilon) > 0$: $|f(x) - A| < \varepsilon$, ha $x \in \underbrace{\dot{K}_\delta(x_0) \cap D_f}_{\{x \in D_f \mid 0 < |x - x_0| < \delta\}}$

b, $\boxed{12}$ $f(x)=\dfrac{|x-2|}{x-2}\cdot\dfrac{\sin(3x)}{x}\cdot\dfrac{1}{x+2}$ A nevező zérushelyei: $-2,\ 0,\ +2$. (2)

f folytonos $\mathbb{R}\setminus\{-2,0,+2\}$-n, mert folyt. fgv-ek hányadosa, és a nevező $\neq 0$. (1)

$x_1=-2$-ben: $$\lim_{x\to-2\pm0} f(x)=\underbrace{\frac{4}{-4}}_{-1}\cdot\underbrace{\frac{\sin(-6)}{-2}}_{>0}\cdot\underbrace{\lim_{x\to-2\pm0}\frac{1}{x+2}}_{\pm\infty}=\mp\infty$$ (3)

$\Rightarrow$ Másodfajú szakadás

$x_2=0$-ban: $$\lim_{x\to0\pm0} f(x)=\frac{2}{-2}\cdot\underbrace{\lim_{x\to0\pm0}3\,\frac{\sin(3x)}{3x}}_{=3}\cdot\frac{1}{2}=-\frac{3}{2}$$ $\Rightarrow$ Megszüntethető szak. (3)

$x_3=+2$-ben: $$\lim_{x\to2\pm0} f(x)=\underbrace{\left(\lim_{x\to2\pm0}\frac{|x-2|}{x-2}\right)}_{\pm1}\cdot\frac{\sin(6)}{2}\cdot\frac{1}{4}=\pm\frac{\sin(6)}{8}$$ $\Rightarrow$ véges ugrás (3)

3, $\boxed{12}$ $$\int_{x=0}^{\infty}\frac{1}{\sqrt{x+1}\,(x+4)}\,dx=\int_1^{\infty}\frac{1}{\cancel{u}\,(u^2-1+4)}\,2\cancel{u}\,du=$$ (5)

$u=\sqrt{x+1}$;

$x=u^2-1$; $dx=2u\,du$

$$=\lim_{\omega\to\infty}\int_1^{\omega}\frac{2}{u^2+3}\,du=\frac{2}{3}\lim_{\omega\to\infty}\int_1^{\omega}\frac{du}{1+\left(\frac{u}{\sqrt3}\right)^2}=$$

$$=\frac{2}{3}\lim_{\omega\to\infty}\left[\operatorname{arctg}\left(\frac{u}{\sqrt3}\right)\cdot\sqrt3\right]_1^{\omega}=\frac{2}{\sqrt3}\lim_{\omega\to\infty}\left(\operatorname{arctg}\frac{\omega}{\sqrt3}-\operatorname{arctg}\frac{1}{\sqrt3}\right)=$$ (5)

$$=\frac{2}{\sqrt3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{2\pi}{3\sqrt3}$$ (2)

$\boxed{12}$ 4, $y(x) = 3e^{2x}\sin(3x) + 16e^{2x} \Rightarrow \lambda_1 = 2;\ \lambda_{2,3} = 2 \pm 3i$ (4)

Kar. pol.: $(\lambda-2)\underbrace{(\lambda-(2+3i))\,(\lambda-(2-3i))}_{(\lambda-2)^2+9} = (\lambda-2)(\lambda^2-4\lambda+13) =$ (3)

$= \lambda^3 - 6\lambda^2 + 21\lambda - 26$

$\Rightarrow \boxed{y''' - 6y'' + 21y' - 26y = 0.}$ (3)

$y_{H,\acute{a}lt}(x) = C_1 e^{2x} + C_2 e^{2x}\sin(3x) + C_3 e^{2x}\cos(3x)$; $C_1, C_2, C_3 \in \mathbb{R}$ (2)

5, a, T.: A adott feltételekkel és jelölésekkel $\exists \xi$ x_0 és x között,

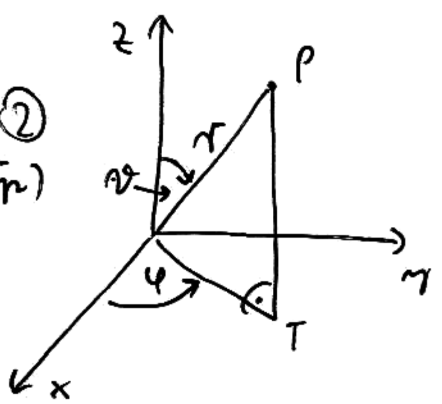
(3) melyre
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$\boxed{5}$ b,
$$\sin\left(\tfrac{1}{2}\right) = \frac{1}{1!}\left(\tfrac{1}{2}\right)^1 - \frac{1}{3!}\left(\tfrac{1}{2}\right)^3 + \frac{1}{5!}\left(\tfrac{1}{2}\right)^5 - \frac{\cos(\xi)}{7!}\left(\tfrac{1}{2}\right)^7$$

$\sin^{(7)}(x) = -\cos x$

Tehát $\sin\left(\tfrac{1}{2}\right) \approx \underbrace{\frac{1}{2} - \frac{1}{48} + \frac{1}{5!\cdot 32}}_{A}$ (3), és $\left|\sin\tfrac{1}{2} - A\right| \le \frac{1}{7!\cdot 2^7}$ (2)

6, a, Gömbi polár rendszer:



(2) (rajz)

$$\left.\begin{aligned} x &= r\sin\vartheta\cos\varphi \\ y &= r\sin\vartheta\sin\varphi \\ z &= r\cos\vartheta \end{aligned}\right\}$$ (3)

4

$$J = \begin{vmatrix} x'_r & x'_\vartheta & x'_\varphi \\ y'_r & y'_\vartheta & y'_\varphi \\ z'_r & z'_\vartheta & z'_\varphi \end{vmatrix} = \begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix} =$$

$$= \cos\vartheta \underbrace{\begin{vmatrix} r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \end{vmatrix}}_{r^2\sin\vartheta\cos\vartheta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}} + r\sin\vartheta \underbrace{\begin{vmatrix} \sin\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \end{vmatrix}}_{r\sin^2\vartheta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}} =$$

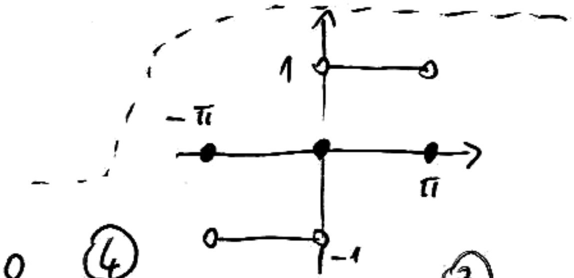
$$= r^2\sin\vartheta\underbrace{(\cos^2\vartheta+\sin^2\vartheta)}_{1} = \underline{\underline{r^2\sin\vartheta}}$$

(Csak a helyes eredmény: 1p)

b) $\boxed{6}$

$$V(R) = \int_{r=0}^{R}\int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi} 1\cdot r^2\sin\vartheta \, d\vartheta\, d\varphi\, dr \overset{3}{=} \left(\int_{r=0}^{R} r^2\,dr\right)\cdot 2\pi\cdot\left(\int_{\vartheta=0}^{\pi}\sin\vartheta\,d\vartheta\right)$$

$$= \frac{R^3}{3}\cdot 2\pi\cdot 2 = \underline{\underline{\frac{4R^3\pi}{3}}} \quad 3$$



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7) $\boxed{13}$ $f(x) = -f(-x)$ páratlan $\Rightarrow$ $a_0 = a_n = 0$ 4

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \overset{2}{=} \frac{2}{\pi}\int_0^{\pi} 1\cdot\sin(nx)\,dx = \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \quad 3$$

$$= \frac{2}{n\pi}\left(-(-1)^n+1\right) = \begin{cases} 0, & \text{ha } n \text{ páros} \\ \frac{4}{n\pi}, & \text{ha } n \text{ páratlan.} \end{cases} \quad 1$$

$\forall x\in\mathbb{R}$-re $f(x) =$ Fourier sor összege, mert az x_0 szakadási helyen is 3

$$f(x_0) = \frac{f(x_0-0)+f(x_0+0)}{2}$$

8, a)

$$\mathcal{F}[f(ax)](\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(\underbrace{ax}_{y=ax})\,dx \overset{\text{ha } a>0}{=} \int_{-\infty}^{\infty} e^{-i\frac{\omega}{a}y} f(y)\,\frac{dy}{a} = \frac{1}{a}\mathcal{F}[f]\left(\frac{\omega}{a}\right)$$

ha $a<0$, akkor $\int_{+\infty}^{-\infty} = -\int_{-\infty}^{+\infty}$, tehát $\forall a\in\mathbb{R}\setminus\{0\}$ esetén $\mathcal{F}[f(ax)](\omega) = \frac{1}{|a|}\mathcal{F}[f]\left(\frac{\omega}{a}\right)$

Kimondás: 3

Biz.: 7

b) $\mathcal{F}^{-1}[F](x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}F(\omega)\,d\omega$ 2

B VARIÁNS (TÖMÖR)

1, a, lásd A; b, A lineár: $\frac{2^3}{3} = \frac{8}{3}$

— ∘ —

2, a, lásd A.

b, $\lim\limits_{x\to -3\pm 0} f(x) = \frac{\sin(-15)}{-3}\cdot\frac{1}{-6}\cdot(\pm 1)$ véges ugrás

$\lim\limits_{x\to 0} f(x) = 5\cdot\frac{3}{-9}$ megszüntethető

$\lim\limits_{x\to +3\pm 0} f(x) = \frac{\sin(15)}{3}\cdot 1\cdot \pm\infty$ másodfajú

— ∘ —

3, $\int_{x=1}^{\infty} \frac{1}{\underbrace{\sqrt{x+2}}_{u}\,(x+11)}\,dx = \int_{u=\sqrt{3}}^{\infty}\frac{1}{\cancel{u}(u^2+9)}\,2\cancel{u}\,du = \frac{2}{9}\lim\limits_{\omega\to\infty}\left[\operatorname{arctg}\frac{u}{3}\cdot 3\right]_{\sqrt{3}}^{\omega} =$

$u;\ x = u^2-2$

$dx = 2u\,du$

$= \frac{2}{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \underline{\underline{\frac{2\pi}{9}}}$

— ∘ —

4, $\lambda_1 = 3,\ \lambda_{2,3} = 3\pm 2i;\quad (\lambda-3)\underbrace{(\lambda-(3+2i))(\lambda-(3-2i))}_{(\lambda-3)^2+4\,=\,\lambda^2-6\lambda+13} = \lambda^3-9\lambda^2+31\lambda-39$

$\Rightarrow \underline{y'''-9y''+31y'-39y=0};\quad \underline{y_{H,\text{ált}}(x) = C_1e^{3x}+C_2e^{3x}\sin(2x)+C_3e^{3x}\cos(2x)}$

— ∘ —

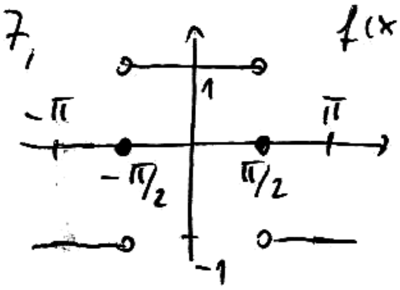
5, a, lásd A, b, $\sin\left(\frac{1}{3}\right) = \underbrace{\frac{1}{3}-\frac{1}{3!}\left(\frac{1}{3}\right)^3+\frac{1}{5!}\left(\frac{1}{3}\right)^5}_{A} - \frac{\cos\xi}{7!}\left(\frac{1}{3}\right)^7;\quad \sin\left(\frac{1}{3}\right)\approx A,\quad H\le\frac{1}{7!\,3^7}$

— ∘ —

6, lásd A,

— ∘ —

7,



$f(x) = \operatorname{sgn}(\cos x)$; f páros $\Rightarrow \forall n: b_n = 0$; $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$;

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{f(x)\cos(nx)}_{\text{prs}}\,dx = \frac{2}{\pi}\left(\int_0^{\pi/2}\cos(nx)\,dx - \int_{\pi/2}^{\pi}\cos(nx)\,dx\right) =$

$= \frac{2}{n\pi}\left(\left[\sin(nx)\right]_0^{\pi/2} - \left[\sin(nx)\right]_{\pi/2}^{\pi}\right) = \frac{4}{n\pi}\sin\left(n\frac{\pi}{2}\right) = \begin{cases} \frac{4}{n\pi}, & \text{ha } n = 4k+1 \\ -\frac{4}{n\pi}, & \text{ha } n = 4k+3 \\ 0, & \text{ha } n \text{ páros} \end{cases}\quad k\in\mathbb{Z}.$

$\forall x\in\mathbb{R}: f(x) = \varphi(x)$

— ∘ —

8, a, $\mathcal{F}\left[f\left(\frac{x}{a}\right)\right](\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f\left(\tfrac{x}{a}\right)dx = |a|\int_{-\infty}^{\infty} e^{-i\omega a y} f(y)\,dy = \underline{|a|\,\mathcal{F}[f](a\omega)}$ $\left(\tfrac{x}{a} = y\right)$

b, lásd A,