2018 január 3, α variáns, mintamegoldás

1. $\boxed{6}$ a) $\displaystyle\int_0^{\frac{\pi}{2}} \sin(2x)\cos^2(2x)\,dx = \frac{1}{2}\int_{-1}^{1} u^2\,du$ 3 $\displaystyle = \frac{1}{6}u^3\Big|_{-1}^{1}$ 2 $\displaystyle = \frac{1}{3}$ 1

$u = \cos 2x,\ du = -2\sin 2x\,dx$

b) $\boxed{6}$ $\displaystyle\int \frac{x+3}{x^2-x-2}\,dx \overset{1}{=} \int \frac{x+3}{(x-2)(x+1)}\,dx = \big(x+3 = A(x+1)+B(x-2)$

$\Rightarrow B = -2/3,\ A = 5/3\big) = \displaystyle\frac{5}{3}\int\frac{dx}{x-2}$ 3 $\displaystyle - \frac{2}{3}\int\frac{dx}{x+1} =$

$\displaystyle = \frac{5}{3}\ln(x-2) - \frac{2}{3}\ln(x+1)$ 2

c) $\boxed{6}$ $\displaystyle\int_{-\infty}^{2} (x+2)e^x\,dx = \lim_{N\to\infty}\int_{-N}^{2}(x+2)e^x\,dx$ 1 $\displaystyle = \lim\Big((x+2)e^x\Big|_{-N}^{2}$

$\displaystyle - e^x\Big|_{-N}^{2}\Big)$ 3 $\displaystyle = \lim_{N\to\infty}\big(4e^2 - (-N+2)e^{-N} - e^2 + e^{-N}\big) = 3e^2$ 2

d) $\boxed{6}$ $\displaystyle\int \frac{x\,dx}{x^2+4x+4} \overset{1}{=} \int\frac{x\,dx}{(x+2)^2} \underset{u=x+2}{=} \int\frac{(u-2)\,du}{u^2}$ 2 $\displaystyle = \ln|u| + \frac{2}{u} + C$

$\displaystyle = \ln|x+2| + \frac{2}{x+2} + C$ 3

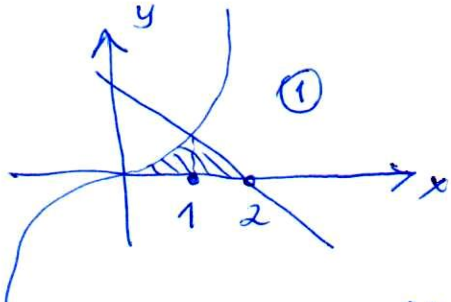
2. $\boxed{2}$ a) f integrálható $[a,b]$-ben és $\exists F' = f$ akkor

$\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ 2

$\boxed{6}$ b) $\displaystyle\int_1^{\infty} x^{-(p+1)}\,dx \overset{p\neq 0}{=} \lim_{N\to\infty} \frac{x^{-p}}{-p}\Big|_N^{\infty} < \infty$ ha $p>0$ 3

3 $p = 0$: $\displaystyle\lim_{N\to\infty} \ln\Big|_N^{\infty} = \infty$. Tehát $\boxed{p>0}$

3. $\boxed{18}$



1

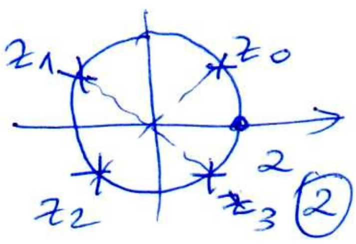
$\displaystyle S = \int_0^1 x^3\,dx + \int_1^2 (2-x)\,dx =$ 2

$\displaystyle = \frac{1}{4}$ 3 $\displaystyle + \Big(2x - \frac{x^2}{2}\Big)\Big|_1^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ 2

4. $\boxed{10}$ $z_k = 2e^{\left(i\pi/4 + \frac{ik\pi}{2}\right)}$, $0 \le k \le 3$ 4

$z_0 = \sqrt{2} + \sqrt{2}i$, $z_1 = -\sqrt{2} + \sqrt{2}i$

$z_2 = -\sqrt{2} - \sqrt{2}i$, $z_3 = \sqrt{2} - \sqrt{2}i$ 4



2

5) $a_n \xrightarrow{n\to\infty} A \Longleftrightarrow \forall \varepsilon>0, \exists n_\varepsilon, |a_n - A| < \varepsilon, n > n_\varepsilon$ 5

$\boxed{10}$ $|\sqrt{a_n} - \sqrt{A}| = \frac{|a_n - A|}{\sqrt{a_n}+\sqrt{A}} \le \frac{|a_n - A|}{\sqrt{A}} \le \frac{\varepsilon}{\sqrt{A}} := \tilde{\varepsilon}$ 5

6) a) $f \in C[a,b] \cap D(a,b) \Longleftrightarrow \exists c \in (a,b)$:

$\boxed{4}$ $f'(c) = \frac{f(b)-f(a)}{b-a}$ 4

b) Ha 3 gyöke lenne $\Rightarrow$ f'-nek 2 $\Rightarrow$ f''-nek 1 gyöke

$\boxed{6}$ van 3. Viszont $f''(x) = 4+3\sin x \ge 1, \forall x$ 3

c) $\lim_{x\to 1}(1+\sin\pi x)^{\operatorname{ctg}\pi x} = \lim_{t\to\pi} e^{\ln(1+\sin t)\operatorname{ctg} t}$ 3

$\boxed{10}$

L'Hôpital: $\lim_{t\to\pi} \frac{\ln(1+\sin t)}{\operatorname{tg} t} = \lim_{t\to\pi} \frac{\frac{1}{1+\sin t}\cdot\cos t}{\frac{1}{\cos^2 t}} = -1$ 5

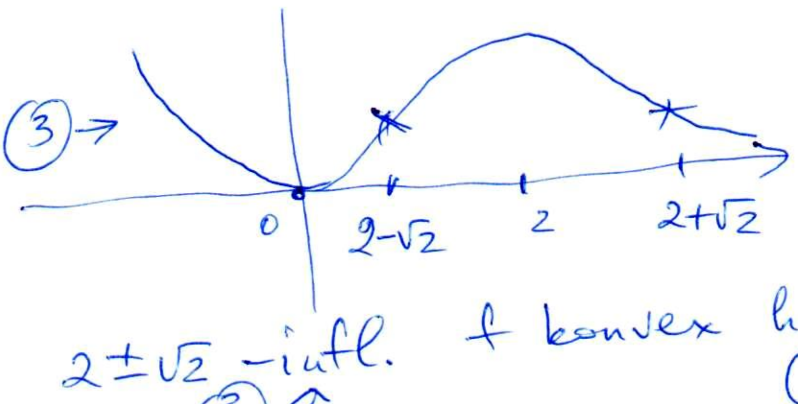
2

7) $f(x) = x^2 e^{-x}$, $f' = (2x - x^2)e^{-x} \begin{cases} <0, & x<0, x>2 \\ \ge 0, & 0 \le x \le 2 \end{cases}$

$\boxed{20}$

$f'' = (2-4x+x^2)e^{-x}$, $f''=0$ ha $x = 2\pm\sqrt{2}$

$ÉT = (-\infty, \infty)$, $ÉK = [0, \infty)$
1 1

3 → 

0 – lok. min 3

$(2, 4e^{-2})$ → lok. max 3

$f\uparrow$, $0\le x\le 2$, $f\downarrow$ $x<0, x>2$ (

$2\pm\sqrt{2}$ – infl. 3 f konvex ha $x<2-\sqrt{2}$, $x>2+\sqrt{2}$ 3

<u>IMSC</u> $\left(\frac{x^2}{x^3+100}\right)' = \frac{x(200-x^3)}{\cdots}$, $x = \sqrt[3]{200}$ – lok. max 7

$5 < \sqrt[3]{200} < 6$. Tehát 5 vagy 6! 5-ben: $\frac{1}{9}$ 6-ban: $9/79 > \frac{1}{9}$

Tehát $\boxed{a_6}$ 7

2018 január 3, β variáns mintamegoldás

1. a) $\int_0^{\frac{\pi}{2}} \sin 2x \cos^3 2x \, dx = -\cos^4 2x / 8 \Big|_0^{\pi/2} = 0$ 3

$\int \sin 2x \cos^3 2x \, dx = -\frac{1}{2}\int u^3 \, du = -\frac{u^4}{8} = -\frac{\cos^4 2x}{8}$ 3

$\cos 2x = u$

b) $\int \frac{x-3}{x^2-x-2} dx \overset{1}{=} \int \frac{x-3}{(x-2)(x+1)} dx = -\frac{1}{3}\int \frac{dx}{x+1} + \frac{4}{3}\int \frac{dx}{x-2}$ 3

$= -\frac{1}{3}\ln|x+1| + \frac{4}{3}\ln|x-2| + C$ 2

c) $\int_{-\infty}^{2} (x+1)e^{2x} dx = \lim_{n\to\infty} \int_{-n}^{2} (x+1)e^{2x} dx = \lim_{\dots} \left(\frac{1}{2}(x+1)e^{2x}\Big|_{-n}^{2} - \frac{1}{4}e^{2x}\Big|_{-n}^{2} \right) = \lim_{n\to\infty}\left(\frac{3}{2}e^4 - \frac{1}{2}(-n+1)e^{-2n} - \frac{1}{4}e^4 + \frac{1}{4}e^{-2n} \right) = \frac{5}{4}e$

1 3 2

d) $\int \frac{x-1}{(x+3)^2} dx \underset{u=x+3}{\overset{1}{=}} \int \frac{u-4}{u^2} du = \ln|u| + \frac{4}{u} = \ln|x+3| + \frac{4}{x+3}$

2 3

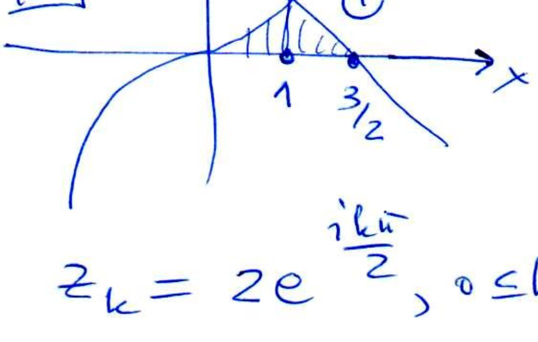
2. a) $f \in R[a,b]$ & $\exists F' = f \Longrightarrow \int_a^b f(x)dx = F(b) - F(a)$

$\boxed{2+6}$ b) $\int_1^\infty x^{-p+1} dx = \lim_{N\to\infty} \int_1^N x^{-p+1} dx = \lim_{N\to\infty} \frac{x^{-p+2}}{-p+2}\Big|_1^N (p \neq$

$< \infty$ ha $p > 2$. Ha $p = 2$: $\ln x \Big|_1^N \to \infty$ $\boxed{p>2}$

3 3

3. $\boxed{8}$



$S = \int_0^1 x^3 dx + \int_1^{3/2} (3-2x) dx$ 2

$= \frac{1}{4} + (3x - x^2)\Big|_1^{3/2} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

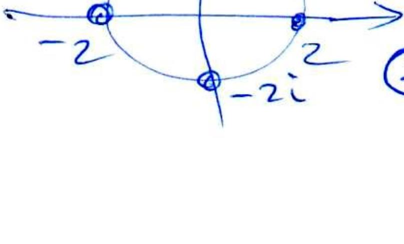
3 2

4. $z_k = 2e^{\frac{ik\pi}{2}}, \ 0 \le k \le 3$ 4

$\boxed{10}$

$z_0 = 2, \ z_1 = 2i, \ z_2 = -2$

$z_3 = -2i$ 4



2

$-2/\beta$

5) $a_n \to A \implies \forall \varepsilon > 0, \exists n_\varepsilon : |a_n - A| < \varepsilon,\ n > n_\varepsilon$ (5)

$\boxed{10}$ $|\sqrt{a_n} - \sqrt{A}| = \dfrac{|a_n - A|}{\sqrt{a_n} + \sqrt{A}} \le \dfrac{|a_n - A|}{\sqrt{A}} \le \dfrac{\varepsilon}{\sqrt{A}} := \tilde{\varepsilon}$ (5), $n > n_\varepsilon$

6) a) $f \in C[a,b] \cap D(a,b) \implies \exists c \in (a,b)$:

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$ (4)

b) Ha 3 gyöke van $\implies$ f'-nek 2 $\implies$ f''-nek 1. (3)

$\boxed{6}$ $f'' = 4 - 3\cos x \ge 1,\ \forall x$ (3)

c) $\lim\limits_{x \to -1} (1 + \sin \pi x)^{\mathrm{ctg}\, \pi x} = \lim\limits_{t \to -\pi} e^{\ln(1+\sin t)\,\mathrm{ctg}\, t}$ (3)

$\boxed{10}$

L'Hôpital: $\lim\limits_{t \to -\pi} \dfrac{\ln(1+\sin t)}{\mathrm{tg}\, t}$ (2) $= \lim\limits_{\dots} \dfrac{\frac{1}{1+\sin t} \cdot \cos t}{\frac{1}{\cos^2 t}} = -1$. (5)

7) $f(x) = x^2 e^x$, $f' = (2x + x^2) e^x \begin{cases} > 0, & x > 0, x < -2 \\ < 0, & -2 < x < 0 \end{cases}$

$\boxed{20}$ $f'' = (2 + 4x + x^2) e^x$, $f'' = 0$ ha $x = -2 \pm \sqrt{2}$

$ÉT = (-\infty, \infty)$ (1) $ÉK = [0, \infty)$ (1)

0 – lok min (3)

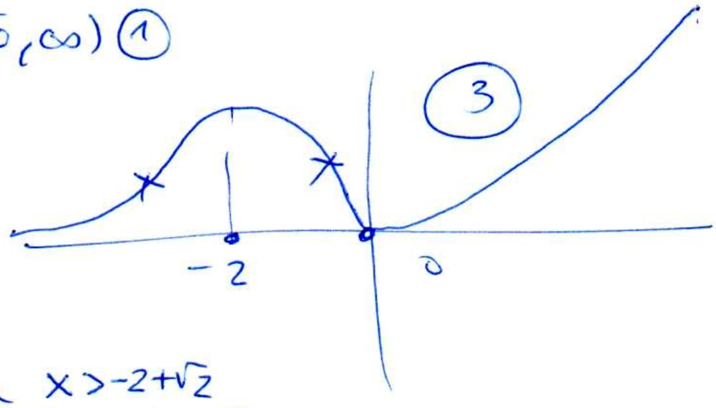
$(-2, 4e^{-2})$ – lok max (3)

$f\uparrow$ ha $x > 0$, $x < -2$ (3)

$f\downarrow$ ha $-2 < x < 0$

$-2 \pm \sqrt{2}$ – infl. (3) f konvex ha $x > -2 + \sqrt{2}$, $x < -2 - \sqrt{2}$ (3)

(3)



IMSC $\boxed{14}$ $\left(\dfrac{x^2}{x^3 + 100} \right)' = \dfrac{x(200 - x^3)}{\dots}$, $x = \sqrt[3]{200}$ – lok. max (7)

$5 < \sqrt[3]{200} < 6$. 5-ben: $1/9$, 6-ban: $9/79$. Tehát $\boxed{a_6}$ (7)