Matematika A3 – 3. vizsga megoldásai

(2015. 01. 14)

① $\frac{y'}{y^3} + \frac{2}{xy^2} = \frac{1}{x^3}$ $\quad y \neq 0$

$u(x) := \frac{1}{y(x)^2} \rightsquigarrow u'(x) = -\frac{2}{y(x)^3} y'(x)$

$\Downarrow$

$\frac{y'}{y^3} = -\frac{1}{2} u'$ $\quad$ 2p

behelyettesítve:

$$-\frac{1}{2}u' + \frac{2}{x}u = \frac{1}{x^3}$$

$$\boxed{u' - \frac{4}{x}u = -\frac{2}{x^3}}$$ 1p

H: $u' - \frac{4}{x}u = 0 \rightsquigarrow \int \frac{du}{u} = \int \frac{4}{x}\,dx$

$\ln|u| = 4\ln|x| + \ln c$

$\Rightarrow \boxed{u_H = cx^4}$ $\quad$ 3p

IH: $u_p(x) = c(x)x^4 \rightsquigarrow u_p'(x) = c'(x)x^4 + 4c(x)x^3$

behelyettesítve: $c'x^4 + 4cx^3 - \frac{4}{x}cx^4 = -\frac{2}{x^3}$

$c'(x) = -\frac{2}{x^7} \rightsquigarrow c(x) = -2\int x^{-7}dx = \frac{1}{3x^6}$

$\Rightarrow \boxed{u_p = \frac{1}{3x^6}\cdot x^4 = \frac{1}{3x^2}}$ 3p $\Longrightarrow u_{ált} = cx^4 + \frac{1}{3x^2} = \frac{1}{y^2}$

$\rightsquigarrow y^2 = \frac{1}{cx^4 + \frac{1}{3x^2}} = \frac{3x^2}{3cx^6+1} \rightsquigarrow \boxed{y_{ált} = \pm\frac{\sqrt{3}\,x}{\sqrt{cx^6+1}}}$ 1p

② Stokes-tétel: $\oint_G \underline{v}(\underline{r})\,d\underline{r} = \iint_F \operatorname{rot} v(\underline{r})\, d\underline{F}$

ahol F olyan mérhető felület, melynek $\partial F = G$

önmhangolt irányítással: $\boxed{4p}$

$\underline{v}(x,y,z) = z^2\underline{i} + x^2\underline{j} + y^2\underline{k}$

$F:\ x^2+y^2+z^2 = 2,\ z \geq 0$ ~> felső gömbhéj

$\Rightarrow G:\ \left.\begin{array}{l} x^2+y^2=2 \\ z=0 \end{array}\right\}$ kör $\Rightarrow$ paraméterezése: $\boxed{2p}$

$$\boxed{\begin{array}{l} \underline{r}(t) = (\sqrt{2}\cos t, \sqrt{2}\sin t, 0) \\ 0 \leq t \leq 2\pi \end{array}}$$

$\rightsquigarrow \dot{\underline{r}}(t) = (-\sqrt{2}\sin t, \sqrt{2}\cos t, 0)$, $\underline{v}(\underline{r}(t)) = 2\cos^2 t\,\underline{j} + 2\sin^2 t\,\underline{k}$

$$\oint_G \underline{v}(\underline{r})\,d\underline{r} = \int_0^{2\pi} \underline{v}(\underline{r}(t))\cdot\dot{\underline{r}}(t)\,dt = \int_0^{2\pi} 2\sqrt{2}\cos^3 t\,dt =$$

$$= 2\sqrt{2}\int_0^{2\pi}\cos t(1-\sin^2 t)\,dt = 2\sqrt{2}\left[\sin t - \frac{\sin^3 t}{3}\right]_0^{2\pi} = \underline{\underline{0}} \quad \boxed{2p}$$

F paraméterezése: $\boxed{\underline{r}(u,v) = (\sqrt{2}\sin u\cos v, \sqrt{2}\sin u\sin v, \sqrt{2}\cos u)}$

$$\boxed{\begin{array}{l} 0 \leq u \leq \frac{\pi}{2} \\ 0 \leq v \leq 2\pi \end{array}}$$

$\hookrightarrow \underline{r}'_u = (\sqrt{2}\cos u\cos v, \sqrt{2}\cos u\sin v, -\sqrt{2}\sin u)$

$\underline{r}'_v = (-\sqrt{2}\sin u\sin v, \sqrt{2}\sin u\cos v, 0)$ $\boxed{2p}$

$$\underline{r}'_u \times \underline{r}'_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \sqrt{2}\cos u\cos v & \sqrt{2}\cos u\sin v & -\sqrt{2}\sin u \\ -\sqrt{2}\sin u\sin v & \sqrt{2}\sin u\cos v & 0 \end{vmatrix} \ (=)$$

② folyt

$$\underline{r}'_u \times \underline{r}'_v \stackrel{\circ}{=} \underline{i}\, 2\sin^2 u\cos v + \underline{j}\, 2\sin^2 u \sin v \; +$$

$$+ \underline{k}\,(2\sin u\cos u\cos^2 v + 2\sin u\cos u\sin^2 v) =$$

$$= (2\sin^2 u\cos v,\, 2\sin^2 u\sin v,\, 2\sin u\cos u)$$

$$\operatorname{rot} \underline{v}(\underline{r}) = \nabla \times \underline{v}(\underline{r}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} =$$

$$= \underline{i}\, 2y + \underline{j}\, 2z + \underline{k}\, 2x \qquad \boxed{1p}$$

$$\hookrightarrow \operatorname{rot} \underline{v}(\underline{r}(u,v)) = (2\sqrt{2}\sin u\sin v,\, 2\sqrt{2}\cos u,\, 2\sqrt{2}\sin u\cos v)$$

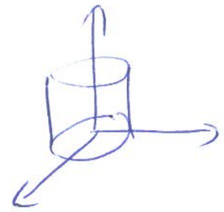
$$\iint_F \operatorname{rot} \underline{v}\, d\underline{F} = \int_0^{2\pi}\int_0^{\pi/2} \underbrace{\operatorname{rot} \underline{v}(\underline{r}(u,v)) \cdot (\underline{r}'_u \times \underline{r}'_v)}_{4\sqrt{2}\left[\sin^3 u\cos v\sin v + \sin^2 u\cos u\sin v + \sin^2 u\cos u\cos v\right]}\, du\, dv \stackrel{\circ}{=}$$

$$\stackrel{\circ}{=} \int_0^{\pi/2}\underbrace{\int_0^{2\pi} 4\sqrt{2}\left[\sin^3 u\cos v\sin v + \sin^2 u\cos u\sin v + \sin^2 u\cos u\cos v\right] dv}_{4\sqrt{2}\left[\sin^3 u\,\frac{\sin^2 v}{2} - \sin^2 u\cos u\cos v + \sin^2 u\cos u\sin v\right]_{v=0}^{2\pi} = 0}\, du$$

$$\Rightarrow \iint_F \operatorname{rot} \underline{v}\, d\underline{F} = \underline{\underline{0}} \qquad \checkmark \qquad \boxed{2p}$$

③ $\underline{v}(x,y,z) = (x-2z)\underline{i} + (2x+y)\underline{j} + (x-y+z)\underline{k}$

$F: \quad \left.\begin{matrix} x^2+y^2=4 \\ 0 \le z \le 2 \end{matrix}\right\}$



zárt henger

$\oiint_F \underline{v}(\underline{r})\,d\underline{F} = ?$

Gauss–Osztrogradszkij

$$\oiint_F \underline{v}(\underline{r})\,d\underline{F} = \iiint_V \operatorname{div} \underline{v}\,dV$$

ahol $\partial V = F$

$\boxed{3p}$

$\operatorname{div} \underline{v} = 1+1+1 = 3$ $\boxed{3p}$

$$\Rightarrow \iiint_V \operatorname{div} \underline{v}\,dV = 3 \underbrace{\iiint_V dV}_{\text{henger térfogata: } r^2\pi \cdot m} = 3 \cdot 2^2 \pi \cdot 2 = \underline{\underline{24\pi}}$$

$\boxed{4p}$

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④ $$\oint_{|z|=2} \left(\frac{1}{(z^4+81)^4} + \frac{\sin z}{z^2(z^2+1)} + z\,e^{-\frac{1}{z^2}} \right) dz$$

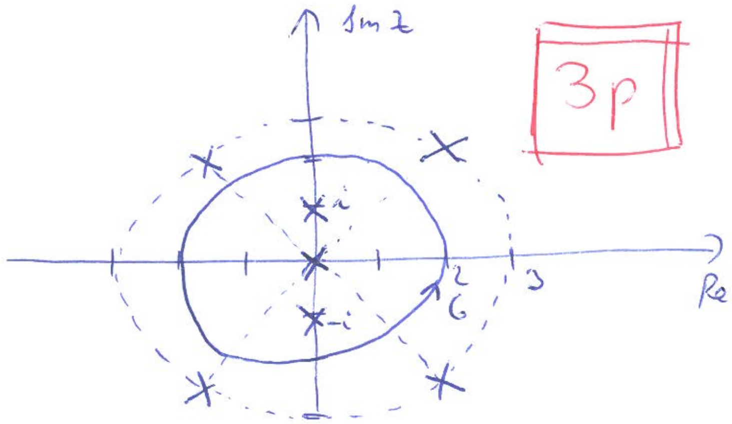
szinguláris helyek:

- $z^4+81=0 \rightsquigarrow z^4 = -81 = 81\,e^{i\pi}$

gyökök: $\boxed{z_k = 3\,e^{i\frac{\pi+2k\pi}{4}} \qquad k=0,1,2,3}$

- $z=0$
- $z^2+1=0 \rightsquigarrow z=\pm i$

csak $z=0, i, -i$ $\Leftarrow$ esik G belsejébe!



$\boxed{3p}$

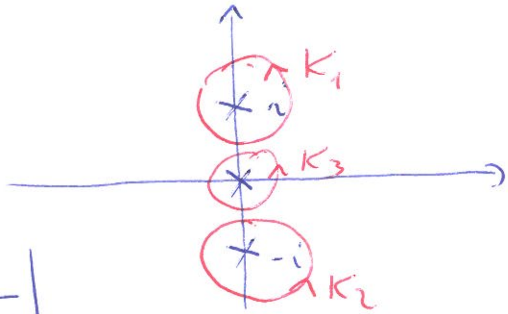
⑤ folyt

$$\Longrightarrow \bullet \oint_G \frac{1}{(z^4+81)^4}\,dz = 0$$, mert G belsejében az integrandus holomorf! (Cauchy-tétel)

$\boxed{2p}$

$$\bullet \oint_{|z|=2} \frac{\sin z}{z^2(z^2+1)}\,dz = \oint_{K_1} \frac{\frac{\sin z}{z^2(z+i)}}{z-i}\,dz + \oint_{K_2} \frac{\frac{\sin z}{z^2(z-i)}}{z+i}\,dz +$$

$$+ \oint_{K_3} \frac{\frac{\sin z}{z^2+1}}{z^2}\,dz \;=$$



$$= \frac{2\pi i}{\cancel{2\pi i}} \frac{\sin z}{z^2(z+i)}\Big|_{z=i} + \frac{2\pi i}{\cancel{2\pi i}} \frac{\sin z}{z^2(z-i)}\Big|_{z=-i}$$

$$+ \frac{2\pi i}{1!}\left(\frac{\sin z}{z^2+1}\right)'\Big|_{z=0} = 2\pi i \frac{\sin i}{-2i} + 2\pi i \frac{\sin(-i)}{2i} +$$

$$+ 2\pi i \frac{\cos z(z^2+1) - \sin z\cdot 2z}{(z^2+1)^2}\Big|_{z=0} = -\pi \sin i + \pi \underbrace{\sin(-i)}_{=-\sin i} + 2\pi i =$$

$$= \underline{\underline{-2\pi \sin i + 2\pi i}}$$ $\boxed{3p}$

$\cancel{+2\pi i}$

$$\left(\cancel{\text{megj}}\quad \sin i = \frac{e^{i\cdot i} - e^{-i\cdot i}}{2i} = \frac{e^{-1}-e}{2i} = \underline{\underline{i\,\mathrm{sh}\,1}}\right)$$

$$\bullet\; e^{-\frac{1}{z^2}} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{z^2}\right)^n}{n!} = 1 - \frac{1}{z^2} + \frac{1}{2!}\cdot\frac{1}{z^4} - \dots$$

$$\rightsquigarrow z e^{-\frac{1}{z^2}} = z - \frac{1}{z} + \frac{1}{2!}\frac{1}{z^3} + \dots \qquad \Longrightarrow \boxed{\operatorname*{Res}_{z=0} z e^{-\frac{1}{z^2}} = -1}$$ $\boxed{2p}$

$$\Longrightarrow \oint_{|z|=2} z e^{-\frac{1}{z^2}}\,dz = 2\pi i(-1) = \underline{\underline{-2\pi i}} \qquad \oint_G \dots = -2\pi i\,\mathrm{sh}\,1 + 2\pi i - 2\pi i = \boxed{\underline{\underline{-2\pi i\,\mathrm{sh}\,1}}}$$

(5)

$\underline{v}(x,y,z) = (2xy - z^{\alpha} - yz)\underline{i} + (x^2 + z^{\alpha} - xz)\underline{j} + (2yz - 2xz - xy)\underline{k}$

$\underline{v}$ potenciálos $\Longleftrightarrow$ rot $\underline{v} = 0$ $\boxed{2p}$

$$\text{rot}\,\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^{\alpha} - yz & x^2 + z^{\alpha} - xz & 2yz - 2xz - xy \end{vmatrix} =$$

$$= \underline{i}\left(2z - x - \alpha z^{\alpha-1} + x\right) - \underline{j}\left(-2z - y + \alpha z^{\alpha-1} + y\right) +$$

$$+ \underline{k}\left(2x - z - 2x + z\right) = \left(2z - \alpha z^{\alpha-1}, 2z - \alpha z^{\alpha-1}, 0\right)$$

rot $\underline{v} = \underline{0}$ $\Longleftrightarrow$ $2z = \alpha z^{\alpha-1}$ $\Longleftrightarrow$ $\boxed{\alpha = 2}$ $\boxed{4p}$

Ekkor $\underline{v}(x,y,z) = (2xy - z^2 - yz, x^2 + z^2 - xz, 2yz - 2xz - xy)$

$\hookrightarrow$ div $\underline{v} = 2y + 0 + 2y - 2x = 4y - 2x$ $\boxed{2p}$

$\Rightarrow$ $\boxed{\text{grad div}\,\underline{v} = (-2, 4, 0)}$ $\boxed{2p}$