ANALÍZIS PRÓBAZH
2001. OKTÓBER 8.
KÓNYA KURZUS

1. feladat (10 pont)
Konvergens-e az a_n sorozat:
$a_1 = 5; \qquad a_{n+1} = \frac{2a_n+3}{5}$

2. feladat (18 pont)
a) $\left(\frac{n^2+5}{n^2-2}\right)^{2n^2+1} = ?$ b) $\left(\frac{n^2+5}{n^2}\right)^{n-2} = ?$

c) $\left(\frac{n^2+5}{n^2}\right)^{n^3} = ?$ d) $\left(1-\frac{2}{n^2}\right)^{n^3} = ?$

3. feladat (9 pont)
a) $\lim\limits_{n\to\infty} \frac{n^2+n+1}{n^3+5n} = ?$

b) $\lim\limits_{n\to\infty} \frac{2n^3+5}{n^3+2\sqrt{n}} = ?$

c) $\lim\limits_{n\to\infty} \frac{n^4+2n+3}{n+2} = ?$

4. feladat (8 pont)

$$\lim_{n\to\infty} \sqrt[n]{3n^2+2n-10} = ?$$

5. feladat (7 pont)
Bizonyítsa be a definíció alapján, hogy $\lim\limits_{n\to\infty} a_n = A$, ha
$a_n = \frac{2n^3+3n+1}{n^3-2n+5}; \qquad A = 2$

6. feladat (8 pont)
Bizonyítsa be, hogy

$$\lim_{n\to\infty} n^3 - 3n^2 + n = \infty$$

7. feladat (8 pont)

$$\lim_{n\to\infty} \left(\sqrt{n^2+3n+1} - \sqrt{n^2-2n+3}\right) =?$$

8. feladat (10 pont)

$$\sum_{k=1}^{\infty} \frac{2^k + 4^{k+2}}{5^{k+1}} =?$$

9. feladat (8 pont)
Konvergens-e az alábbi sor?

$$\sum_{n=1}^{\infty} \frac{2n+3}{(n+1)!}$$

10. feladat (14 pont)

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n^2+1}$$

Becsülje meg a hibát, ha $s \approx s_{100}$!

KÓNYA

$\boxed{10}$

1. • monoton csökken mert:

$a_2 = \frac{13}{5} \quad a_2 < a_1$

tfh. $a_{n+1} < a_n$

B be, hogy $a_{n+2} \overset{?}{<} a_{n+1}$

$$2a_{n+1} < 2a_n$$
$$2a_{n+1} + 3 < 2a_n + 3$$
$$\frac{2a_{n+1}+3}{5} < \frac{2a_n+3}{5} \quad 3$$
$$a_{n+2} < a_{n+1} \ \checkmark$$

• lehetséges határ: $A = \frac{2A+3}{5} \to A = 1$ 2

• alulról korlátos mert:

$a_1 > 1$

tfh. $a_n > 1$

B be, hogy $a_{n+1} \overset{?}{>} 1$

$$2a_n > 2$$
$$2a_n + 3 > 5$$
$$\frac{2a_n+3}{5} > 1$$
$$a_{n+1} > 5 \ \checkmark \quad 3$$

konvergens (mert m. csökken és alulról 2 korlátos)

$\Downarrow$

$\lim_{n\to\infty} a_n = 1$

2. $\boxed{4}$ a, $\left(\frac{n^2+5}{n^2-2}\right)^{2n^2+1} = \left(\left(\underset{\downarrow\, e^7}{1+\frac{7}{n^2-2}}\right)^{n^2-2}\right)^{2}\ \underbrace{\left(\underset{\downarrow\, 0}{1+\frac{7}{n^2-2}}\right)^{5}}_{\downarrow\, 1} \to \cancel{e^{2}} \ e^{14}$

← véges hatv. szám

$\boxed{5}$ b, $\left(\frac{n^2+5}{n^2}\right)^{n-2} = \left(\frac{n^2+5}{n^2}\right)^{-2}\left(\frac{n^2+5}{n^2}\right)^{n} = \left(\underset{\downarrow\, 1}{1+\frac{5}{n^2}}\right)^{-2} \cdot \sqrt[n]{\left(1+\frac{5}{n^2}\right)^{n^2}} \to 1$

$\cancel{\left(1+\frac{5}{n^2}\right)^{n^2} \to e^5 \to 1}$ ROSSZ

$$\underset{\downarrow\, 1}{\sqrt[n]{2^5}} \le \sqrt[n]{\left(1+\frac{5}{n^2}\right)^{n^2}} \le \underset{\downarrow\, 1}{\sqrt[n]{3^5}}$$

(rendőrelv)

$\boxed{4}$ c, $\left(\left(1+\frac{5}{n^2}\right)^{n^2}\right)^{n} > \left(2^5\right)^n \to \infty$

$\boxed{5}$ d, $\left(\left(1-\frac{2}{n^2}\right)^{n^2}\right)^{n} \le \left(\frac{1}{2}\right)^{2n} \to 0$

$\cancel{(e^{-2})}$

$\left(\frac{1}{e} < \frac{1}{2}\right)$

(3) a, $\boxed{3}$ $\dfrac{\cancel{n^2}(1+\frac{1}{n}) + \frac{1}{n^2}}{\cancel{n^2}(n+\frac{5}{n})} = \left(\frac{1}{\infty}\right) = \emptyset$

b, $\boxed{3}$ $\dfrac{\cancel{n^3}(2+\frac{5}{n^3})}{\cancel{n^3}(1+\frac{2}{n^3})} = 2$

$\frac{5}{2}$

c, $\boxed{3}$ $\dfrac{\cancel{n}(n^3+2+\frac{3}{n})}{\cancel{n}(1+\frac{2}{n})} = \infty$

(4) $\boxed{8}$ $\sqrt[n]{3n^2} < \sqrt[n]{3n^2+n} \le \sqrt[n]{3n^2+2n-10} \le \sqrt[n]{3n^2+2n^2} = \sqrt[n]{5n^2}$

$\sqrt[n]{3}\,\sqrt[n]{n}\,\sqrt[n]{n}$ $\to 1, 1, 1$ $\qquad$ $\sqrt[n]{5}\,\sqrt[n]{n}\,\sqrt[n]{n}$ $\to 1, 1, 1$

$\to 1 \leftarrow$

rendőrelv

(5) $\boxed{7}$ $|a_n - A| = \left|\dfrac{2n^3+3n+1}{n^3-2n+5} - 2\right| = \left|\dfrac{2n^3+3n+1-2n^3+4n-10}{n^3-2n+5}\right| = \dfrac{7n-9}{n^3-2n+5} \le$ ($n>2$)

$\le \dfrac{7n}{n^3-2n} \le \dfrac{7n}{\frac{1}{2}n^3} = \dfrac{14}{n^2} < \varepsilon$

$N(\varepsilon) \ge \sqrt{\dfrac{14}{\varepsilon}}$

(6) $\boxed{8}$ $n^3 - 3n^2 + 5 > n^3 - 3n^2 = \frac{1}{2}n^3 + \frac{1}{2}n^3 - 3n^2 > \frac{1}{2}n^3 > M$

$n^2(\frac{1}{2}n - 3) \ge 0$ $\qquad$ $n > \sqrt[3]{2M}$

$n \ge 6$ $\longrightarrow \max(6, \sqrt[3]{2M})$

(7) $\boxed{8}$

$\dfrac{\cancel{n^2}+3n+1-\cancel{n^2}+2n-3}{\sqrt{n^2+3n+1}+\sqrt{n^2-2n+3}} = \dfrac{5n-2}{\sqrt{n^2+3n+1}+\sqrt{n^2-2n+3}} = \dfrac{\cancel{n}(5-\frac{2}{n})}{\sqrt{n^2}\left(\sqrt{1+\frac{3}{n}+\frac{1}{n^2}}+\sqrt{1-\frac{2}{n}+\frac{3}{n^2}}\right)}$

$= \dfrac{5}{2}$

(8) $\boxed{10}$ $s_n = \sum_{k=1}^{n}\left(\dfrac{2^k}{5^{k+1}} + \dfrac{4^{k+2}}{5^{k+1}}\right) = \sum_{k=1}^{n} \frac{1}{5}\left(\frac{2}{5}\right)^k + \frac{16}{5}\left(\frac{4}{5}\right)^k = \frac{1}{5}\sum_{k=1}^{n}\left(\frac{2}{5}\right)^k + \frac{16}{5}\sum_{k=1}^{n}\left(\frac{4}{5}\right)^k$

$\Rightarrow \dfrac{1}{5}\,\underbrace{\dfrac{1}{1-\frac{2}{5}}}_{\frac{1}{1-q}}\,\underbrace{\frac{2}{5}}_{a_1} + \dfrac{16}{5}\,\underbrace{\dfrac{1}{1-\frac{4}{5}}}_{\frac{1}{1-q}}\,\underbrace{\frac{4}{5}}_{a_1}$

$q^k \quad |q|<1 \to$ összeg $\dfrac{1}{1-q}$

⑨ [8] Hányados kritérium

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2n+7}{(n+2)!}\cdot\frac{\cancel{(n+1)!}}{2n+3}=\lim_{n\to\infty}\frac{2n+7}{2n^2+7n+6}=$$

$$=\lim_{n\to\infty}\frac{\cancel{n}(2+\frac{7}{n}\to 0)}{\cancel{n}(2n+7+\frac{6}{n}\to 0)}=0<1 \Rightarrow \text{konvergens}$$

(igen)

⑩ [14] Leibniz típusú (2) • alternáló $(-1)^{n+1}$ ✓ (2)

• $\lim_{n\to\infty}|a_n|=\emptyset$

$$\lim_{n\to\infty}\frac{2n+1}{n^2+1}=\lim_{n\to\infty}\frac{\cancel{n}(2+\frac{1}{n}\to 0)}{\cancel{n}(n+\frac{1}{n}\to 0)}=\emptyset$$ ✓ (3)

• mon. csökken

$$a_{n+1}\overset{?}{<}a_n$$

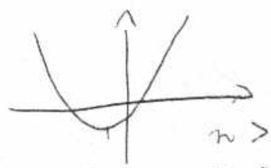
$$\frac{2(n+1)+1}{(n+1)^2+1}\overset{?}{<}\frac{2n+1}{n^2+1}$$

$$\frac{2n+3}{n^2+2n+2}\overset{?}{<}\frac{2n+1}{n^2+1}$$

$$\cancel{2n}+3+\cancel{2n^3}+3n^2<\cancel{2n^3}+4n^2+4n+n^2+\cancel{2n}+2$$

$$0<2n^2+4n-1$$

$$\frac{-4\pm\sqrt{24}}{4}=\begin{cases}(\approx 0{,}225) & -1+\frac{\sqrt{24}}{2\sqrt{6}}\\ (\approx -2{,}225) & -1-\sqrt{24}\end{cases}$$



$n>0$ -nál

$2n^2+4n-1>0$

$\Downarrow$

$a_{n+1}<a_n$ (4) ✓

$\Downarrow$

Leibniz típusú

$$|H|\le c_{101}=\frac{2\cdot 101+1}{(101)^2+1}$$ (3)

(a hiba az első figyelembe nem vett tag)