α VARIÁNS: (Részletes)

1, **D.:** a_n Cauchy-sorozat, ha $\forall \varepsilon > 0$ esetén $\exists N(\varepsilon) \in \mathbb{N}$, hogy

$\boxed{13}$ (3) $\forall n, m > N(\varepsilon)$ esetén $|a_n - a_m| < \varepsilon$.

T.: Ha $\exists \lim_{n\to\infty} a_n = A$, akkor a_n Cauchy-sorozat.

(6) **B.:** Legyen adott $\varepsilon > 0$. Mivel $a_n \to A$, ezért $\exists N(\varepsilon) \in \mathbb{N}$:
$\forall n > N(\varepsilon)$: $|a_n - A| < \varepsilon$. De akkor
$\forall n, m > N(\varepsilon)$: $|a_n - a_m| = |(a_n - A) - (a_m - A)| \le |a_n - A| + |a_m - A| < 2\varepsilon$

(2) Az állítás megfordítása $\mathbb{Q}$-ban nem igaz, Pl.:
$\underbrace{1;\ 1.4;\ 1.41;\ 1.414;\ 1.4142;}_{\sqrt{2} \text{ közelítő tizedes törtjei}, \in \mathbb{Q}} \ldots \longrightarrow \sqrt{2} \notin \mathbb{Q}$

(2) Az állítás megfordítása $\mathbb{R}$-ben igaz.

2, a, **D.:** • f az I-n konvex, ha $\forall a, b \in I$ esetén $\forall x \in [a,b]$-re

$\boxed{15}$ (3) $f(x) \le h(x)$, ahol h az a, b közti húr

(bármelyik def. jó)

• ~, ha $\forall a, b \in I$, $\forall \lambda \in [0,1]$: $f(\lambda a + (1-\lambda) b) \le \lambda f(a) + (1-\lambda) f(b)$

• ~, ha $\forall a, b \in I$ esetén $f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}$

b, **T.:** Ha f kétszer diff.-ható I-n, és $f''(x) \ge 0$ $\forall x \in I$, akkor
(3) f konvex I-n.

$\boxed{9}$ c, $f(x) = (1+3x^2)^{-\frac{1}{2}}$; $f'(x) = -\frac{1}{2} \cdot (1+3x^2)^{-3/2} \cdot 6x = -3x(1+3x^2)^{-3/2}$ (2)

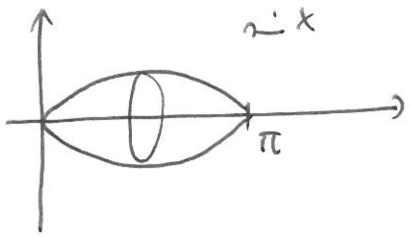
$f''(x) = -3(1+3x^2)^{-3/2} - 3x\left(-\frac{3}{2}\right)(1+3x^2)^{-5/2} \cdot 6x = \frac{-3(1+3x^2) + 27x^2}{(1+3x^2)^{5/2}} =$ (2)

$= \frac{3(\sqrt{6}x + 1)(\sqrt{6}x - 1)}{(1+3x^2)^{5/2}}$ (1)

(4)

x	$x < \frac{-1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$	$1/\sqrt{6}$	$\frac{1}{\sqrt{6}} < x$
f''	+	0	−	0	+
f	$\cup$	infl. p.	$\cap$	infl. p.	$\cup$

3, $\boxed{10}$



$$V = \pi \int_a^b f^2(x)\,dx = \pi \int_0^{\pi} \sin^2 x\,dx = \quad (5)$$

$$= \pi \int_0^{\pi} \frac{1-\cos(2x)}{2}\,dx = \pi\left(\frac{\pi}{2} - \frac{1}{2}\underbrace{\int_0^{\pi}\cos(2x)\,dx}_{0}\right) = \frac{\pi^2}{2} \quad (5)$$

4, $\boxed{10}$ (H) $\lambda^2 - 6\lambda + 10 = 0$; $\lambda_{12} = 3 \pm \sqrt{3^2-10} = 3 \pm i$ (4)

$y_{H,\text{ált}}(x) = A e^{3x}\sin x + B e^{3x}\cos x$, $A, B \in \mathbb{R}$

NINCS rezonancia; $y_{I,p}(x) = C e^{3x}$, bírva!

$9C\cancel{e^{3x}} - 6\cdot 3C\cancel{e^{3x}} + 10C\cancel{e^{3x}} = 5\cancel{e^{3x}} \Rightarrow (9-18+10)C = 5 \Rightarrow C = 5$ (5)

$y_{I,p}(x) = 5e^{3x}$; $y_{I,\text{ált}}(x) = y_{H,\text{ált}}(x) + y_{I,p}(x) = A e^{3x}\sin x + B e^{3x}\cos x + 5e^{3x}$ (1)

5, $\boxed{15}$ a, Ha $[a,b]$-n $\sum_{n=1}^{\infty} f_n \rightrightarrows S$ (egyenletes konvergencia), és

(4) $\forall n$: $f_n \in R[a,b]$, akkor $S \in R[a,b]$, és $\int_a^b S(x)\,dx = \sum_{n=1}^{\infty}\int_a^b f(x)\,dx$)

$\boxed{4}$ b, $\boxed{11}$ $S(x) = \sum_{n=0}^{\infty}(n+2)x^n$; $\sqrt[n]{1} \le \sqrt[n]{n+2} \le \sqrt[n]{3n} \Rightarrow R = \frac{1}{1} = 1.$ (2)

$\downarrow 1 \quad \downarrow 1 \quad \downarrow 1$

$x = \pm 1$-ben $\sum_{n=0}^{\infty}(n+2)(\pm 1)^n = \nexists$, mert az ált. tag $\not\to 0$. (2)

Tehát K.T. $= (-1, +1)$

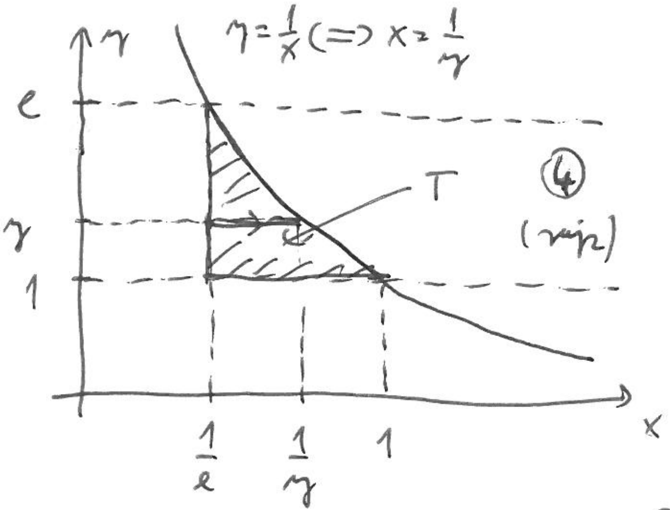
$\boxed{7}$ Összeg: $x \cdot S(x) = \sum_{n=0}^{\infty}(n+2)x^{n+1}$;

$$\int_0^x t\,S(t)\,dt = \int_0^x\left(\sum_{n=0}^{\infty}(n+2)t^{n+1}\right)dt = \sum_{n=0}^{\infty}(n+2)\overbrace{\int_0^x t^{n+1}\,dt}^{\frac{x^{n+2}}{n+2}} = \sum_{n=0}^{\infty}x^{n+2} = \frac{x^2}{1-x}, \text{ ha } |x|<1 \quad (4)$$

$$x\,S(x) = \left(\frac{x^2}{1-x}\right)' = \frac{2x(1-x) - x^2(-1)}{(1-x)^2} = \frac{x(2-x)}{(1-x)^2}; \quad S(x) = \frac{2-x}{(1-x)^2}, \text{ ha } |x| < 1. \quad (3)$$

6, $\boxed{12}$

$$I=\int_{y=1}^{e}\int_{x=1/e}^{1/y}\cos(x-\ln x)\,dx\,dy=$$



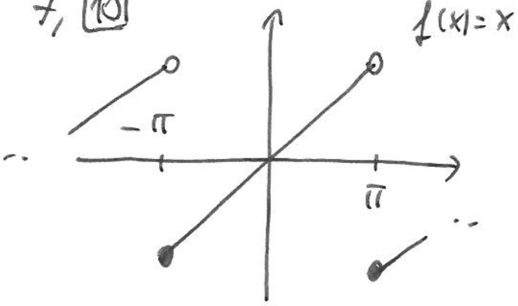
$$=\iint_T \cos(x-\ln x)\,dT=$$

$$=\int_{x=\frac{1}{e}}^{1}\int_{y=1}^{1/x}\cos(x-\ln x)\,dy\,dx \;③\; =\int_{x=\frac{1}{e}}^{1}\cos(x-\ln x)\left(\frac{1}{x}-1\right)dx \;①\; =\Big[-\sin(x-\ln x)\Big]_{1/e}^{1} \;②\; =$$

$-\cos(\varphi(x))\,\varphi'(x)$ alak

$$=-\sin(1)+\sin\left(\frac{1}{e}+1\right) \quad ②$$

7*, $\boxed{10}$



$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}x\,dx=0 \quad ②$$ (∫-ok felírása: 1-1 pont.)

$$a_3=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{x\cos(3x)}_{\text{páratlan}}\,dx=0 \quad ③$$

$$b_5=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{x\sin(5x)}_{\text{páros}}\,dx=\frac{2}{\pi}\int_{0}^{\pi}x\sin(5x)\,dx=\frac{2}{\pi}\underbrace{\left[-x\,\frac{\cos(5x)}{5}\right]_{0}^{\pi}}_{\frac{\pi}{5}-0}+\frac{2}{\pi}\underbrace{\int_{0}^{\pi}1\cdot\frac{\cos(5x)}{5}\,dx}_{0}=\frac{2}{5} \quad ⑤$$

8*

a, ④ $(f*g)(x)=\int_{t=-\infty}^{\infty}f(t)\,g(x-t)\,dt$ $\left(\text{vagy } \int_{-\infty}^{\infty}f(x-t)\,g(t)\,dt\right)$

b, ⑦ $(f*g)(x)=\int_{t=-\infty}^{\infty}f(t)\,g(\underbrace{x-t}_{y})\,dt=\int_{y=+\infty}^{-\infty}f(x-y)\,g(y)\,(-dy)=\int_{y=-\infty}^{\infty}g(y)\,f(x-y)\,dy=$

$=(g*f)(x)$ ✓

c, ⑥ $\mathcal{F}[f*g]=\mathcal{F}[f]\cdot\mathcal{F}[g]$

(B VARIÁNS: (Júnior)

1, Lásd α-variáns 3+6+4 = 13 p.

2, (3) a, f konkáv, ha $\forall\, a, b \in I,\ \lambda \in [0,1]$: $f(\lambda a + (1-\lambda) b) \geq \lambda f(a) + (1-\lambda) f(b)$

(3) b, $f''(x) \leq 0 \quad \forall x \in I \Rightarrow f$ konkáv I-n.

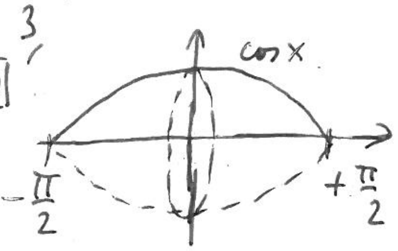
$\boxed{9}$ c, $f'(x) = -5x(1+5x^2)^{-3/2}$ (2) ; $f''(x) = -5(1+5x^2)^{-3/2} + 75x^2(1+5x^2)^{-5/2} =$ (2)

$= 5(1+5x^2)^{-5/2}(\sqrt{10}\,x+1)(\sqrt{10}\,x-1)$ (1)

x		$-1/\sqrt{10}$		$1/\sqrt{10}$	
f''	+	0	−	0	+
f	∪	infl. p.	∩	infl. p.	∪

(4)

3, $\boxed{10}$



$V = \pi \int_{-\pi/2}^{\pi/2} \cos^2 x\, dx = \pi \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2x)}{2}\, dx = \frac{\pi^2}{2}$ (5) (5)

4, $\boxed{10}$ (H): $\lambda^2 + 4\lambda + 5 = (\lambda+2+i)(\lambda+2-i) = 0 \Rightarrow \lambda_{12} = -2 \pm i$

$y_{H,ált}(x) = Ae^{-2x}\sin x + Be^{-2x}\cos x$ (4)

(I): $y_{I,p}(x) = Ce^{-2x}$; $(4 + 4(-2) + 5)C = 3 \Rightarrow C = 3$; $y_{I,p} = 3e^{-2x}$ (5)

$y_{I,ált}(x) = Ae^{-2x}\sin x + Be^{-2x}\cos x + 3e^{-2x}$ (1)

5, a, Lásd α. (4)

$\boxed{11}$ b, K.T.: $(-1,+1)$ (4) ; $\int_{t=0}^{x} t^2 S(t)\, dt = \sum_{n=0}^{\infty} (n+3) \underbrace{\int_0^x t^{n+2}\, dt}_{\frac{x^{n+3}}{n+3}} = \sum_{n=0}^{\infty} x^{n+3} = \frac{x^3}{1-x}$, ha $|x|<1$ (4)

$S(x) = \frac{1}{x^2}\left(\frac{x^3}{1-x}\right)' = \frac{3x^2(1-x)+x^3}{x^2(1-x)^2} = \frac{3-2x}{(1-x)^2}$, ha $|x|<1$ (3)

6, Lásd α. (12)

7, $a_0 = 0$ (2) ; $a_5 = 0$ (3) ; $b_3 = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(3x)\, dx = \frac{2}{\pi}\left[\frac{-x\cos(3x)}{3}\right]_0^{\pi} + \frac{2}{3\pi}\int_0^{\pi} \cos(3x)\, dx =$

$= \frac{2}{3}$ (5)

8, Lásd α, 4+7+4 = 15 p.