12.8 Exercises

1. Find the Z transform of the following functions:

   (a) n^3, (b) $\frac{a^n}{n!}$, (c) $n\ \exp\{(n-1)\alpha\}$,

   (d) $H(n) - H(n-2)$, (e) $n^2 a^n$, (f) $\delta(n) = \left\{\begin{matrix} 1, & n=0, \\ 0, & \text{otherwise} \end{matrix}\right\}$.

2. Show that

   (a) $$Z\{\sinh\ na\} = \frac{z\,(\sinh\ a)}{z^2 - 2z\ \cosh a + 1},$$

   (b) $$Z\{\exp(-an)\cos bn\} = \frac{z\,(z - e^{-a}\cos b)}{z^2 - 2\,z\,e^{-a}\cos b + e^{-2a}}.$$

   (c) $$Z\left\{e^{-an}\sin\ bn\right\} = \frac{e^a\ z\ \sin\ b}{e^{2a}\ z^2 - 2\,e^a\ z\cos\ b + 1}, \qquad |z| > e^{-a}.$$

3. Show that

   $$Z\{n\,a^n f(n)\} = -z\frac{d}{dz}\left\{F\left(\frac{z}{a}\right)\right\}.$$

4. Prove that

   (a) $$Z\left\{\frac{f(n)}{n}\right\} = \int_z^\infty \frac{F(z)}{z}\,dz,$$

   (b) $$Z\left\{\frac{f(n)}{n+m}\right\} = z^m \int_z^\infty \frac{F(z)\,dz}{z^{m+1}}.$$

Hence, deduce that

$$Z\left\{\frac{1}{n+1}\right\} = z\log\left(\frac{z}{z-1}\right).$$

5. Show that

   (a) $Z\{na^{n-1}\} = \dfrac{z}{(z-a)^2}$,

   (b) $Z\left\{\dfrac{n(n-1)\cdots(n-m+1)}{m!}\,a^{n-m}\right\} = \dfrac{z}{(z-a)^{m+1}}$.

6. Find the inverse Z transform of the following functions:

   (a) $\dfrac{z^2}{(z-2)(z-3)}$, (b) $\dfrac{z^2-1}{z^2+1}$, (c) $\dfrac{z}{(z-1)^2}$,

   (d) $\dfrac{z}{(z-a)^2}$, (e) $\dfrac{1}{(z-a)^2}$, (f) $\dfrac{1}{(z-1)^2(z-2)}$,

   (g) $\dfrac{z+3}{(z+1)(z+2)}$, (h) $\dfrac{z^3}{(z^2-1)(z-2)}$, (i) $\dfrac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$.

   (j) $F(z) = \frac{z^2}{(z-e^{-a})(z-e^{-b})}$, a, b are constants.

   (k) $F(z) = (z-a)^{-k}$, $k = 1, 2, \ldots$, $|z| > |a| > 0$.

   (l) $F(z) = \frac{z^4+5}{(z-1)^2(z-2)}$, $|z| > 2$, (m) $F(z) = \frac{(z-1)}{(z+2)(z-\frac{1}{2})}$, $|z| > 2$.

7. Solve the following difference equations:

   (a) $f(n+1) + 3f(n) = n$, $f(0) = 1$.

   (b) $f(n+1) - 5f(n) = \sin n$, $f(0) = 0$.

   (c) $f(n+1) - af(n) = a^n$, $f(0) = x_0$.

   (d) $f(n+1) - f(n) = a[1 - f(n)]$, $f(0) = x_0$.

   (e) $f(n+2) - f(n+1) - 6f(n) = 0$, $f(0) = 0$, $f(1) = 3$.

   (f) $f(n+2) + 4f(n+1) + 3f(n) = 0$, $f(0) = 1$, $f(1) = 1$.

   (g) $f(n+2) - f(n+1) - 6f(n) = \sin\left(\frac{n\pi}{2}\right)$ $(n \geq 2), f(0) = 0$, $f(1) = 3$.

   (h) $f(n+2) - 2f(n+1) + f(n) = 0$, $f(0) = 2$, $f(1) = 0$.

   (i) $f(n+2) - 2af(n+1) + a^2 f(n) = 0$, $f(0) = 0$, $f(1) = a$.

   (j) $f(n+3) - f(n+2) - f(n+1) + f(n) = 0$, $f(0) = 1, f(1) = f(2) = 0$.

   (k) $f(n) = f(n-1) + 2f(n-2)$, $f(0) = 1, f(1) = 2$.

   (l) $f(n) - a\,f(n-1) = 1$, $f(-1) = 2$.

(m) $f(n+2)+3\,f(n+1)+2\,f(n)=0,\ f(0)=1,\ f(1)=2.$

(n) $f(n+1)-2\,f(n)=0,\quad f(0)=3.$

8. Show that the solution of the resistive ladder network governed by the difference equation for the current field $i(n)$

$$i(n+2)-3\,i(n+1)+i(n)=0,\quad i(0)=1,\quad i(1)=2\,i(0)-\frac{V}{R}$$

is

$$i(n)=\cosh(xn)+\frac{2}{\sqrt{5}}\left(\frac{1}{2}-\frac{V}{R}\right)\sinh(nx),$$

where $\cosh x=\dfrac{3}{2}$ and $\sinh x=\dfrac{\sqrt{5}}{2}$.

9. Use the Initial Value Theorem to find $f(0)$ for $F(z)$ given by

(a) $\dfrac{z}{z-\alpha}$, (b) $\dfrac{z}{(z-\alpha)(z-\beta)}$,

(c) $\dfrac{z(z-\cos x)}{z^2-2z\cos x+1}$, (d) $\dfrac{1}{(z-a)^m}$.

10. Use the Final Value Theorem to find $\lim_{n\to\infty} f(n)$ for $F(z)$:

(a) $F(z)=\dfrac{z}{z-a}$, (b) $F(z)=\dfrac{z^2-z\cos a}{(z^2-2z\cos a+1)}$.

11. Find the sum of the following series using the Z transform:

(a) $\displaystyle\sum_{n=0}^{\infty} a^n e^{inx}$, (b) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{e^{-n}}{n+1}$, (c) $\displaystyle\sum_{n=0}^{\infty}\exp[-x(2n+1)]$.

12. Solve the second order difference equation

$$3f(n+2)-2f(n+1)-f(n)=0,\quad f(0)=1,\quad f(1)=2$$

and then show that $f(n)\to\dfrac{7}{4}$ as $n\to\infty$.

13. Solve the simultaneous difference equations

$$\begin{aligned}u(n+1)&=2\,v(2)+2,\\ v(n+1)&=2\,u(n)-1,\end{aligned}$$

with the initial data $u(0)=v(0)=0$.

14. Show that the solution of the third order difference equation

$$u(n+3) - 3\,u(n+2) + 3\,u(n+1) - u(n) = 0,$$
$$u(0) = 1,\ u(1) = 0,\ u(2) = 1,$$

is

$$u(n) = (n-1)^2.$$

15. Show that the solution of the initial value problem

$$u(n+2) - 4\,u(n+1) + 3\,u(n) = 0, \quad u(0) = u_0 \text{ and } u(1) = u_1$$

is

$$u_n = \frac{1}{2}(3u_0 - u_1) + \frac{1}{2}(u_1 - u_0)3^n.$$

16. Find the solution of the following initial value problems:

(a) $u_{n+2} + 2\,u_{n+1} - 3\,u_n = 0, \quad u_0 = 1, \quad u_1 = 0,$

(b) $3\,u_{n+2} - 5\,u_{n+1} + 2\,u_n = 0, \quad u_0 = 1, \quad u_1 = 0,$

(c) $u_{n+2} - 4\,u_{n+1} + 5\,u_n = 0, \quad u_0 = \frac{1}{2}, \quad u_1 = 3.$