1. $\dfrac{dy}{dx} = \dfrac{\sqrt{y^2+1}}{x^2-1}$, $y(2)=0$.

(Nincs konstans megoldás) 1

$\int \frac{1}{\sqrt{y^2+1}}\, dy = \int \frac{1}{x^2-1}\, dx$ 5

$\operatorname{arsinh} y = -\operatorname{artanh} x + C$ 5

$C = \operatorname{artanh} 2.$ 2

2 $y(x) = \sinh(\operatorname{artanh} 2 - \operatorname{artanh} x)$

2. $y' + 2(x+1)y = e^{-x^2}$

(a) $y' + 2(x+1)y = 0$

$y_h(x) = c\, e^{-(x+1)^2}$, $c \in \mathbb{R}$ 6

(b) $y(x) := c(x)\, e^{-(x+1)^2}$ 2

$c'(x)\, e^{-(x+1)^2} = e^{-x^2}$ 4

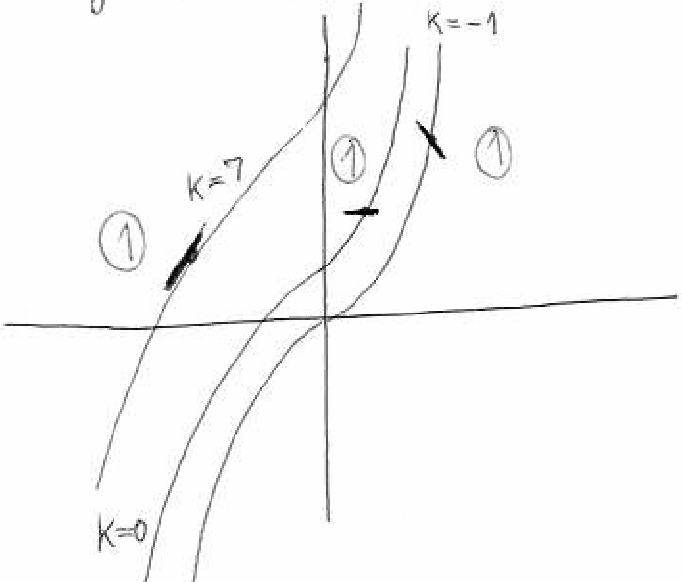
$c(x) = \dfrac{e^{2x+1}}{2}$

(c) 2 $y(x) = c\, e^{-(x+1)^2} + \dfrac{e^{\overbrace{2x+1-(x+1)^2}^{}_{=-x^2}}}{2}$

4. $y' = (y - \sinh(x))^3 - 1$

(a) $(y - \sinh(x))^3 - 1 = K$

$y = \sinh(x) + \sqrt[3]{K+1}$ 4



(b) $x_0 = 0$, $y_0 = 1$

$y'(0) = 0$ 1

$y''(x) = 3(y(x) - \sinh(x))^2 (y'(x) - \cosh(x))$ 4

$y''(0) = 3 \cdot 1 \cdot (-1) < 0$ 2

lok. max. 1

3. $y' = (x+2y)^2$, $y\left(\frac{1}{\sqrt{2}}\right) = 0.$

$u(x) := x + 2y(x).$ 2

$y = \dfrac{u-x}{2}.$ 1

$u' - 1 = 2u^2$, 2 $\dfrac{du}{dx} = 1 + 2u^2.$ 1

$\int \frac{1}{1+2u^2}\, du = \int 1\, dx$,

$\dfrac{\arctan(\sqrt{2}u)}{\sqrt{2}} = x + c.$ 5

$u\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$, $\dfrac{\pi/4}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} + c$,

$c = \dfrac{\pi/4 - 1}{\sqrt{2}}.$ 2

2

$\arctan(\sqrt{2}x + 2\sqrt{2}y) = \sqrt{2}x + \frac{\pi}{4} - 1.$

⑥ (a) $\sum_{n=0}^{\infty} \frac{n!}{n^n}$

$\frac{n!}{n^n} = \frac{n(n-1)\ldots 2\cdot 1}{n \; n \quad n \; n} \le \frac{2}{n^2}$, $n \ge 3$.

$\sum \frac{2}{n^2} < \infty \Rightarrow$ majoráns kritérium $\sum_{n=0}^{\infty} \frac{n!}{n^n} < \infty$.

(b) $\sum_{n=0}^{\infty} \left(\frac{2+3n}{5+3n}\right)^{n^2}$

gyökkritérium

$\lim \left(\frac{2+3n}{5+3n}\right)^n = \lim \left\{ \left[\frac{\left(1+\frac{2}{3n}\right)}{\left(1+\frac{5}{3n}\right)} \right]^{3n} \right\}^{1/3} = e^{\frac{2-5}{3}} = e^{-1} < 1$

konvergens.

(c) $\sum_{n=0}^{\infty} \frac{1}{\sqrt[n]{n^3+7}}$

$\sqrt[n]{n^3} \le \sqrt[n]{n^3+7} \le \sqrt[n]{2n^3}$, $n \ge 2$.

$\searrow \; 1 \; \swarrow \Rightarrow \lim \frac{1}{\sqrt[n]{n^3+7}} \; 1 \ne 0.$

⑦ $f(n+2) = -f(n+1) + 6f(n)$, $f(0) = -2$, $f(1) = 21$

$f(n) := q^n$. ②

$q^2 + q - 6 = 0 \Rightarrow q_1 = 2, \; q_2 = -3.$ ②

$f_h(n) = c_1 2^n + c_2 (-3)^n$ ③

$\left.\begin{array}{l} f(0) = c_1 + c_2 = -2 \\ f(1) = 2c_1 - 3c_2 = 21 \end{array}\right\}$ $c_1 = 3$, $c_2 = -5$ ②

$\underline{f(n) = 3\cdot 2^n - 5\cdot(-3)^n}$. ①

⑤ $y'''(x) - y''(x) - 2y'(x) = 3x + 5e^x$.

① $\lambda^3 - \lambda^2 - 2\lambda = 0 \Rightarrow \lambda_1 = 0, \; \lambda_2 = 2, \; \lambda_3 = -1$ ③

$y_h(x) = c_1 + c_2 e^{2x} + c_3 e^{-x}$, $c_1, c_2, c_3 \in \mathbb{R}$. ①

próbafüggvény: $y_1(x) = A e^x \Rightarrow A = -\frac{5}{2}$ ③

$y_2(x) = Ax^2 + Bx \Rightarrow A = -\frac{3}{4}, \; B = \frac{3}{4}$. ⑤

$\boxed{y(x) = c_1 + c_2 e^{2x} + c_3 e^{-x} - \frac{5}{2} e^x - \frac{3}{4} x^2 + \frac{3}{4} x, \quad c_1, c_2, c_3 \in \mathbb{R}}$ ②