Info anal 2, VD1, MINTAMEGOLDÁS -1- 2014.05.29.

1, $y'' + y' - 2y = 3xe^x$

(H): $\lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1) = 0 \Rightarrow \lambda_1 = -2;\ \lambda_2 = +1$

$$y_{H,ált}(x) = C_1 e^{-2x} + C_2 e^x \quad (5)$$

(I): Külső rezonancia van!

$x(-2)/\quad y_{I,p}(x) = x(Ax+B)e^x = (Ax^2+Bx)e^x \quad (2)$

$x(1)/\quad y_{I,p}'(x) = (2Ax+B)e^x + (Ax^2+Bx)e^x = (Ax^2+(2A+B)x+B)e^x$

$x(1)/\quad y_{I,p}''(x) = (2Ax+2A+B)e^x + (Ax^2+(2A+B)x+B)e^x =$

$(+)\quad = (Ax^2+(4A+B)x+2A+2B)e^x$

$$3xe^x = e^x(\cancel{-2Ax^2} \cancel{-2Bx} + \cancel{Ax^2} + (2A+\cancel{B})x + B + \cancel{Ax^2} + (4A+\cancel{B})x + 2A + 2B)$$

$$3xe^x = (\underbrace{6A}_{3}x + \underbrace{2A+3B}_{0})e^x \qquad 6A = 3 \Rightarrow A = \frac{1}{2}$$

$$2A + 3B = 0 \Rightarrow B = -\frac{1}{3}$$

$$y_{I,p}(x) = \left(\frac{1}{2}x^2 - \frac{1}{3}x\right)e^x \quad (3)$$

$$y_{I,á}(x) = y_{H,á}(x) + y_{I,p}(x) = C_1 e^{-2x} + C_2 e^x + \left(\frac{x^2}{2} - \frac{x}{3}\right)e^x \quad (2)$$

2, $y' = \frac{1}{2}(4y^2 + 4xy + x^2 + 3) = \frac{1}{2}(\underbrace{(2y+x)}_{u(x)}^2 + 3) \qquad /\cdot 2$

(3)

$u(x) = 2y(x) + x;\ u' = 2y' + 1;\ 2y' = u' - 1$

$u' - 1 = u^2 + 3;\quad u' = u^2 + 4 \quad (2)$

$$\int \frac{du}{u^2+4} = \int dx \Rightarrow \frac{1}{4}\int \frac{du}{1+\left(\frac{u}{2}\right)^2} = x + C$$ ②

$$\frac{1}{4}\,\mathrm{arctg}\left(\frac{u}{2}\right)\cdot 2 = x + C \;;\; \frac{u}{2} = \mathrm{tg}\,(2(x+C)) \qquad (u = 2y + x)$$ ③

$$\underline{\underline{y_{ált}(x) = \mathrm{tg}\,(2(x+C)) - \frac{x}{2}}}$$ ②

3, a, Legyen $a_n \geq 0$ $\forall n \in \mathbb{N}$ esetén.

i, Ha $\forall n \in \mathbb{N}$: $\sqrt[n]{a_n} \leq c < 1$, akkor $\sum_n a_n$ konvergens.

ii, Ha $\forall n \in \mathbb{N}$: $\sqrt[n]{a_n} \geq 1$, akkor $\sum_n a_n$ divergens. ③

(Lehet $\forall n > N_0 \in \mathbb{N}$ is, ill. ii.-ben végtelen sok n-re)

$\boxed{6}$ b, i, $\sqrt[n]{a_n} \leq c \Rightarrow a_n \leq c^n$, így $\sum_n a_n \leq \sum_n c^n < \infty$, ③ ha $|c| < 1$.

ii, $\sqrt[n]{a_n} \geq 1 \Rightarrow a_n \geq 1 \Rightarrow \sum_n a_n = \infty$ (nem teljesül a konv. szükséges feltétele.) ③

$\boxed{6}$ c, $$\sum_{n=1}^{\infty} \underbrace{\left(\frac{3n+4}{3n+7}\right)^{n^2+n}}_{a_n}$$

$$\sqrt[n]{a_n} = \left(\frac{3n+4}{3n+7}\right)^{n+1} = \underbrace{\left(\frac{3n+4}{3n+7}\right)}_{\to 1} \cdot \frac{\overbrace{\left(1+\frac{4/3}{n}\right)^n}^{\to e^{4/3}}}{\underbrace{\left(1+\frac{7/3}{n}\right)^n}_{\to e^{7/3}}} \xrightarrow{n\to\infty} \frac{e^{4/3}}{e^{7/3}} = e^{-1} < 1$$

Tehát $\sum_{n=1}^{\infty} a_n$ konvergens.

4, a, Az f fgv. x_0 bázispontú Taylor-sora:

[3] $$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

b, Az I véges intervallumon $T(x) = f(x)$, ha $\{f^{(n)}\}_{n\in\mathbb{N}}$-ek

[3]

egyenletesen korlátosak I-n, azaz $\exists\, K \in \mathbb{R}$:

$$\forall x \in I,\ \forall n \in \mathbb{N}:\ |f^{(n)}(x)| < K.$$

c, A Lagrange-féle maradéktagot kell becsülnünk:

[6] $$|f(x) - T_n(x)| = |R_n(x)| = \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}\right| \le$$

$$\le \frac{K}{(n+1)!} R^{n+1} \xrightarrow{n\to\infty} 0, \quad \text{ahol } R = \sup_{x\in I}\{|x-x_0|\}$$

5, [9] $$f(x) = (8+5x^2)^{-1/3} = \frac{1}{2}\left(1+\frac{5x^2}{8}\right)^{-1/3} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-1/3}{n}\left(\frac{5}{8}\right)^n x^{2n}$$ (5)

ha $\left|\frac{5x^2}{8}\right| < 1$, azaz $|x| < \sqrt{\frac{8}{5}} = R$ (2)

$$f^{(6)}(0) = 6!\cdot a_6 = 6!\cdot\frac{1}{2}\binom{-1/3}{3}\left(\frac{5}{8}\right)^3 = \frac{6!}{2}\cdot\frac{(-1/3)(-4/3)(-7/3)}{3!}\cdot\left(\frac{5}{8}\right)^3$$ (2)

$2n=6,\ n=3$

6,* a, Szükséges feltétel: Ha f-nek (x_0,y_0)-ban lokális szélsőértéke van, akkor $f'_x(x_0,y_0) = f'_y(x_0,y_0) = 0$. (3)

(4) Elégséges feltétel: Ha $\operatorname{grad} f(x_0,y_0) = 0$, és $\begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy}\end{vmatrix}(x_0,y_0) > 0$, akkor f-nek lokális szélsőértéke van (x_0,y_0)-ban.
Ha $f''_{xx}(x_0,y_0) > 0$, akkor lok. min., ha $f''_{xx}(x_0,y_0) < 0$, akkor lok. max.

6b, [9]

$$f'_x(x,y) = 3x^2 - 3y = 0 \Rightarrow x^2 = y$$
$$f'_y(x,y) = 3y^2 - 3x = 0 \Rightarrow y^2 = x$$
$$\Rightarrow y^4 = y,$$

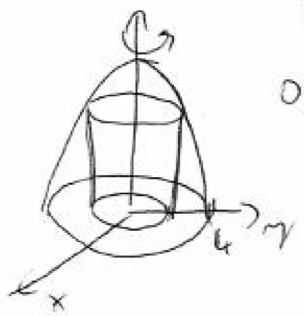
$$\left.\begin{array}{l} y_1 = 0,\ x_1 = 0 \\ y_2 = 1,\ x_2 = 1 \end{array}\right\}$$ Lehetséges szélsőérték helyek. (2)

$$|\underline{\underline{H}}(x,y)| = \begin{vmatrix} f''_{xx}(x,y) & f''_{xy}(x,y) \\ f''_{yx}(x,y) & f''_{yy}(x,y) \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$$ (3)

(0,0)-ban $|\underline{\underline{H}}(0,0)| = -9 < 0 \Rightarrow$ nincs lok. szélsőérték (nyeregpont) (2)

(1,1)-ben $|\underline{\underline{H}}(0,0)| = 36 - 9 = 27 > 0$, és $f''_{xx}(1,1) = 6 > 0 \Rightarrow$ lokális minimum (2)

7, [10]



$x^2 + y^2 \le 4$

$0 \le z \le 16 - (x^2 + y^2)$

Henger koordinátákkal:

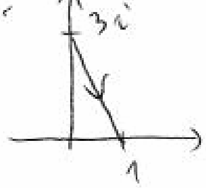
$$\left.\begin{array}{l} 0 \le r \le 2 \\ 0 \le z \le 16 - r^2 \\ 0 \le \varphi \le 2\pi \end{array}\right\}$$ (3)

$$V = \int_{\varphi=0}^{2\pi} \int_{r=0}^{2} \int_{z=0}^{16-r^2} 1 \cdot r \, dz\, dr\, d\varphi =$$ (2)

$$= 2\pi \int_{r=0}^{2} r(16 - r^2)\, dr = 2\pi \left[8r^2 - \frac{r^4}{4}\right]_0^2 = 64\pi - 8\pi = \underline{\underline{56\pi}}$$ (5)

8a, [5]

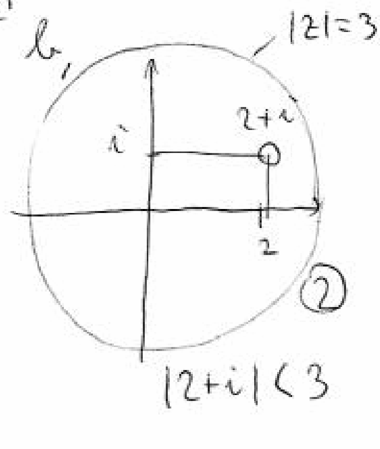
L:



$$\int_L \cos(2z)\, dz = \left[\frac{\sin(2z)}{2}\right]_{3i}^{1} = \frac{\sin 2 - \sin(6i)}{2} =$$ (3)

$$= \underline{\underline{\frac{\sin 2}{2} - \frac{\mathrm{sh}\, 6}{2} i}}$$ (2)

9b, [9]



(2)

$|2+i| < 3$

$$\oint_{|z|=3} \frac{\mathrm{sh}(z)}{(z-(2+i))^2}\, dz = \frac{2\pi i}{1!} \mathrm{sh}'(2+i) = 2\pi i\, \mathrm{ch}(2+i) =$$ (4)

$n = 1$

$z_0 = 2 + i$

$$= 2\pi i\,(\mathrm{ch}\,2\, \underbrace{\mathrm{ch}\, i}_{\cos 1} + \mathrm{sh}\,2\, \underbrace{\mathrm{sh}\, i}_{i \sin 1}) =$$

$$= \underline{\underline{-2\pi\, \mathrm{sh}\,2 \cdot \sin 1 + 2\pi i\, \mathrm{ch}\,2 \cdot \cos 1}}$$ (3)