1/

# A3 2. ZH - Megoldókulcs

(1) $y'' - 3y' + 2y = 3e^{2x}$ $\quad y(0) = -1 \;,\; y'(0) = 3$

H: $y'' - 3y' + 2y = 0 \rightsquigarrow$ kar. polinom: $\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0$

$\hookrightarrow \lambda_1 = 2 \;,\; \lambda_2 = 1$

$\Rightarrow$ homogén ált. mo: $\boxed{y_{H\acute{a}}(x) = C_1 e^{2x} + C_2 e^{x}}$ $\boxed{5p}$

IH: határozatlan együtthatók módszere: $\boxed{3p}$

$y_p(x) = A e^{2x} \cdot x$ ← külső rezonancia

$\hookrightarrow y_p'(x) = 2Ae^{2x} \cdot x + Ae^{2x} = Ae^{2x}(2x+1)$

$y_p''(x) = 2Ae^{2x}(2x+1) + 2Ae^{2x}$

behelyettesítve: $\cancel{4Axe^{2x}} + 4Ae^{2x} - \cancel{6Axe^{2x}} - 3Ae^{2x} + \cancel{2Ae^{2x} \cdot x} = 3e^{2x}$

$Ae^{2x} = 3e^{2x} \Rightarrow \underline{\underline{A = 3}}$

inhomogén partikuláris mo: $\boxed{y_{ip}(x) = 3e^{2x}}$ $\boxed{5p}$

ált. mo: $\boxed{y_{ált}(x) = C_1 e^{2x} + C_2 e^{x} + 3e^{2x}}$ $\boxed{2p}$

$y(0) = C_1 + C_2 + 3 = -1$

$y'(x) = 2C_1 e^{2x} + C_2 e^{x} + 6e^{2x} \rightsquigarrow y'(0) = 2C_1 + C_2 + 6 = 3$

$\Big\}$ $\boxed{C_1 = 1 \;,\; C_2 = -5}$

$\underline{\underline{y(x) = 4e^{2x} - 5e^{x}}}$

$\boxed{5p}$

(2

(2)

megoldás $\Rightarrow$ kar. polinom gyöke gyök multipl.

- $x^2 \Rightarrow \lambda = 0$ 3-szoros $\boxed{2p}$
- $xe^{-x} \Rightarrow \lambda = -1$ 2-szeres $\boxed{2p}$
- $\sin 3x \Rightarrow \lambda = \pm 3i$ 1-szeres $\boxed{2p}$

$\hookrightarrow$ kar. polinom: $\lambda^3(\lambda+1)^2(\lambda-3i)(\lambda+3i) =$

$$= \lambda^3 \underbrace{(\lambda^2+2\lambda+1)(\lambda^2+9)}_{\lambda^4+2\lambda^3+10\lambda^2+18\lambda+9} =$$

$$= \lambda^7 + 2\lambda^6 + 10\lambda^5 + 18\lambda^4 + 9\lambda^3$$ $\boxed{2p}$

$\Downarrow$

a diffegyenlet:

$$\boxed{y^{(7)} + 2y^{(6)} + 10y^{(5)} + 18y^{(4)} + 9y''' = 0}$$

$\boxed{2p}$

3

$$\dot{x}(t) = x(t) - 5y(t) \qquad x(0) = -6$$

$$\dot{y}(t) = 2x(t) - y(t) \qquad y(0) = 0$$

Laplace-trafóval:

$$x(t) \overset{\mathcal{L}}{\longmapsto} X(s)$$

$$y(t) \longmapsto Y(s)$$

$$\dot{x}(t) \longmapsto sX(s) - x(0) = sX(s) + 6$$

$$\dot{y}(t) \longmapsto sY(s) - y(0) = sY(s)$$

$$\Longrightarrow \begin{cases} sX + 6 = X - 5Y & (1.) \\ sY = 2X - Y & (2) \end{cases} \qquad \boxed{5p}$$

$$(2) \rightsquigarrow \quad Y = \frac{2}{s+1} X$$

$$(1.) \rightsquigarrow \quad sX + 6 = \cancel{X} - \frac{10}{s+1} X$$

$$X \underbrace{\left(s - 1 + \frac{10}{s+1}\right)}_{\frac{s^2+9}{s+1}} = -6 \quad \Longrightarrow \quad \boxed{X = -6 \, \frac{s+1}{s^2+9}} \qquad \boxed{3p}$$

$$\Downarrow$$

$$\boxed{Y = \frac{-12}{s^2+9}} \qquad \boxed{3p}$$

③ folyt.

$$X(s) = -6\,\frac{s+1}{s^2+9} = -6\left(\frac{s}{s^2+9}\right) - \frac{6}{3}\left(\frac{3}{s^2+9}\right)$$

$$\left(\frac{s}{s^2+9}\right) \xrightarrow{\mathcal{L}^{-1}} \cos 3t \qquad \left(\frac{3}{s^2+9}\right) \xrightarrow{\mathcal{L}^{-1}} \sin 3t$$

inverz Laplace:

$$\frac{s}{s^2+\omega^2} \xmapsto{\mathcal{L}^{-1}} \cos \omega t$$

$$\frac{\omega}{s^2+\omega^2} \xmapsto{\mathcal{L}^{-1}} \sin \omega t$$

$$\Rightarrow \boxed{x(t) = -6\cos 3t - 2\sin 3t} \qquad \boxed{5p}$$

$$Y(s) = -\frac{12}{3}\,\frac{3}{s^2+9} \xmapsto{\mathcal{L}^{-1}} \boxed{y(t) = -4\sin 3t} \qquad \boxed{4p}$$

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④ $u(x,y) = cx^2 + 2xy - 4y^2 + 3$

Mikor lesz harmonikus?

$$u'_x(x,y) = 2cx + 2y$$

$$u''_{xx}(x,y) = 2c$$

$$u'_y(x,y) = 2x - 8y$$

$$u''_{yy}(x,y) = -8$$

$$\Delta u = u''_{xx} + u''_{yy} = 2c - 8 = 0 \iff \boxed{c = 4} \qquad \boxed{6p}$$

5) folyt

C-R egyenletek:

$$u'_x = v'_y = 2x - 8y$$ $\boxed{4p}$

$$u'_y = -v'_x = -8x - 2y$$

$\hookrightarrow$ $$u(x,y) = \int (2x-8y)\,dx = x^2 - 8xy + C(y)$$

$$\downarrow$$

$$u'_y(x,y) = -8x + \frac{dC(y)}{dy} = -8x - 2y$$

$$\Downarrow$$

$$\frac{dC(y)}{dy} = -2y \implies C(y) = \int (-2y)\,dy = -y^2 + c$$

$$\boxed{u(x,y) = x^2 - 8xy - y^2 + c}$$ $\boxed{5p}$

$$\implies \boxed{f(x,y) \equiv f(z) = x^2 - 8xy - y^2 + c + i(4x^2 + 2xy - 4y^2 + 3)}$$

$\boxed{2p}$

$$f'(z) = u'_x(x,y) - i u'_y(x,y) = (2x - 8y) + i(8x - 2y)$$

$$\Downarrow$$

$$f'(1-2i) = 2 + 16 + i(8+4) = \underline{\underline{18 + 12i}}$$

$\boxed{3p}$

5

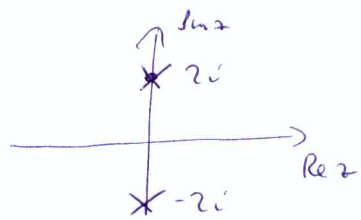
$$\oint_G \underbrace{\frac{z^4 e^{\pi z}}{z^2+4}}_{= f(z)}\,dz$$

számláló és nevező is mindenütt holomorf,

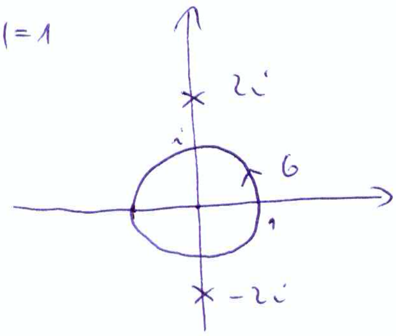
↪ szinguláris helyek = nevező zérushelyei:

$\boxed{3p}$

$$z^2+4=(z-2i)(z+2i)=0 \iff z=2i,\quad z=-2i$$




a) $G: |z|=1$



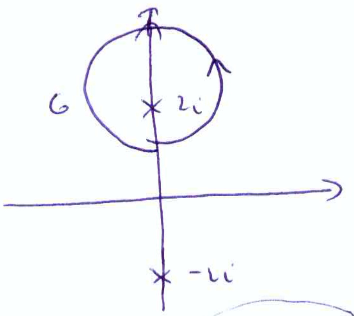

G belsejében $f(z)$ holomorf, így a Cauchy-tétel miatt

$$\oint_G f(z)\,dz = 0$$

$\boxed{7p}$

b) $G: |z-2i|=1$

G csak a $2i$ pólust tartalmazza a belsejében




holomorf G belsejében

$$\Rightarrow \oint_G f(z)\,dz = \oint_G \frac{\frac{z^4 e^{\pi z}}{z+2i}}{z-2i}\,dz \underset{\uparrow}{=} 2\pi i \left.\frac{z^4 e^{\pi z}}{z+2i}\right|_{z=2i} =$$

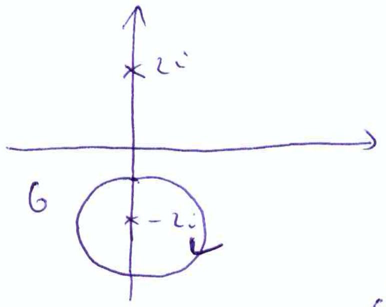
Cauchy I. int. fm

$$= 2\pi i \,\frac{16 i^4 \overbrace{e^{2\pi i}}^{=1}}{4i} = \underline{\underline{8\pi}}$$

$\boxed{10p}$

(5) folyt.

c) $G: |z+2i| = 1$ G belseje csak a $-2i$ pólust tartalmazza




holomorf G belsejében

$$\oint_G f(z)\,dz = - \oint_{-G} \frac{\frac{z^4 e^{\pi z}}{z-2i}}{z+2i}\,dz =$$

neg. irányítás

$$= -2\pi i \left. \frac{z^4 e^{\pi z}}{z-2i} \right|_{z=-2i} = -2\pi i \frac{16 i^4 e^{-2\pi i}}{-4i} = \underline{\underline{8\pi}}$$

Cauchy I. intform..

$\boxed{10p}$