A3 2. vizsgadolgozat (2015.01.07.) – Megoldások

① $y''' - 2y'' + y' = x + 2e^x \quad , \; y(0) = 5 \;, \; y'(0) = 3 \;, \; y''(0) = 4$

<u>H</u>: $y''' - 2y'' + y' = 0 \;\rightsquigarrow$ kar. polinom: $\lambda^3 - 2\lambda^2 + \lambda =$

$$= \lambda(\lambda - 1)^2 = 0$$

$\hookrightarrow \lambda_1 = 0 \;, \; \lambda_2 = \lambda_3 = 1 \;\rightsquigarrow$ belső rezonancia

$$\Rightarrow \boxed{y_H(x) = C_1 + C_2 e^x + C_3 x e^x}$$

3p

<u>IH</u>: próbafv $\rightsquigarrow$ külső rezonancia miatt:

$$y_p = (Ax + B)\cdot x + Ce^x \cdot x^2 = Ax^2 + Bx + Ce^x \cdot x^2$$

$\hookrightarrow \; y_p' = 2Ax + B + Ce^x(x^2 + 2x)$

2p

$y_p'' = 2A + Ce^x(x^2 + 4x + 2)$

$y_p''' = Ce^x(x^2 + 6x + 6)$

behelyettesítve:

$$Ce^x(x^2 + 6x + 6) - 2\left(2A + Ce^x(x^2 + 4x + 2)\right) + 2Ax + B + Ce^x(x^2 + 2x) = x + 2e^x$$

$\hookrightarrow \; 2Ce^x + 2Ax + B - 4A = x + 2e^x \;\rightsquigarrow \boxed{C = 1 \;, \; A = \frac{1}{2} \;, \; B = 2}$

ált. mo: $\boxed{y_{ált}(x) = C_1 + C_2 e^x + C_3 x e^x + \frac{1}{2}x^2 + 2x + e^x \cdot x^2}$ 3p

$y(0) = C_1 + C_2 = 5$

$y' = C_2 e^x + C_3 e^x(x + 1) + x + 2 + e^x(x^2 + 2x) \rightsquigarrow y'(0) = C_2 + C_3 + 2 = 3$

$y'' = C_2 e^x + C_3 e^x(x + 2) + 1 + e^x(x^2 + 4x + 2) \rightsquigarrow y''(0) = C_2 + 2C_3 + 3 = 4$

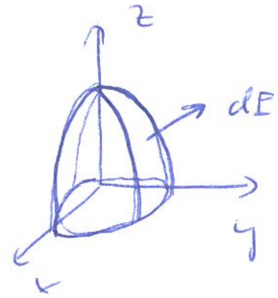
$$\left.\begin{aligned} C_1 + C_2 &= 5 \\ C_2 + C_3 &= 1 \\ C_2 + 2C_3 &= 1 \end{aligned}\right\} \Rightarrow C_3 = 0 \;, \; C_2 = 1 \;, \; C_1 = 4$$

$$\boxed{y(x) = 4 + e^x + \frac{1}{2}x^2 + 2x + e^x \cdot x^2}$$

2p

(2) $\underline{v}(x,y,z) = x^2y^2\,\underline{i} + \underline{j} + z\,\underline{k}$

$\mathcal{F}:$ $\left.\begin{matrix} z = 1-x^2-y^2 \\ z = 0 \end{matrix}\right\}$ által határolt zárt felület



Gauss-Osztrogradszkij tétellel:

$$\oiint_{\mathcal{F}} \underline{v}\, d\underline{F} = \iiint_V \operatorname{div} \underline{v}\, dV \;\textcircled{=}$$

$\operatorname{div} \underline{v} = 2xy^2 + 1$

V paraméterezése: forgásszimmetria

$\Downarrow$

hengerkoordináták

$\boxed{\|2p}$

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

↳ ha $z=0 \rightsquigarrow x^2+y^2=1$

$z = 1-r^2$ a határfelületen

$$\begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \\ 0 \le z \le 1-r^2 \end{cases}$$

$\boxed{\|3p}$

$$\textcircled{=} \int_0^{2\pi}\int_0^1\int_0^{1-r^2} (2r\cos\varphi \cdot r^2\sin^2\varphi + 1)\cdot \underset{\substack{\uparrow \\ \text{Jacobi-det}}}{r}\, dz\, dr\, d\varphi =$$

$\boxed{\|1p}$

$$= \int_0^{2\pi}\int_0^1 \underbrace{\int_0^{1-r^2} (2r^4\sin^2\varphi\cos\varphi + r)\, dz}\, dr\, d\varphi =$$

$$(2r^4\sin^2\varphi\cos\varphi + r)\left[z\right]_0^{1-r^2} = 2r^4\sin^2\varphi\cos\varphi + r - 2r^6\sin^2\varphi\cos\varphi - r^3$$

$$= \int_0^{2\pi} \left[2\frac{r^5}{5}\sin^2\varphi\cos\varphi + \frac{r^2}{2} - 2\frac{r^7}{7}\sin^2\varphi\cos\varphi - \frac{r^4}{4}\right]_0^1 d\varphi =$$

$$= \left(\frac{2}{5}-\frac{2}{7}\right)\overset{\overset{0}{\shortparallel}}{\left[\frac{\sin^3\varphi}{3}\right]_0^{2\pi}} + \left(\frac{1}{2}-\frac{1}{4}\right)\left[\varphi\right]_0^{2\pi} = \frac{1}{4}\cdot 2\pi = \underline{\underline{\frac{\pi}{2}}}$$

$\boxed{\|4p}$

(3) $\underline{v}(x,y,z) = (3x^2y - y^3)\,\underline{i} + (x^3 - 3xy^2)\,\underline{j}$

$$\operatorname{rot} \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y - y^3 & x^3 - 3xy^2 & 0 \end{vmatrix} = \underline{i}\cdot 0 - \underline{j}\cdot 0 + \underline{k}\left(3x^2 - 3y^2 - 3x^2 + 3y^2\right) = 0$$

$\Downarrow$

mindenütt potenciálos

2p

$\Rightarrow \; \exists \; U = U(x,y,z)$ potenciálfv, melyre

$$\operatorname{grad} U = \underline{v} \iff \left.\begin{aligned} \frac{\partial U}{\partial x} &= 3x^2y - y^3 \\ \frac{\partial U}{\partial y} &= x^3 - 3xy^2 \\ \frac{\partial U}{\partial z} &= 0 \end{aligned}\right\}$$

2p

$\hookrightarrow \; U(x,y,z) = \int (3x^2y - y^3)\,dx = x^3y - xy^3 + C(y,z)$

$\hookrightarrow \; \dfrac{\partial U}{\partial y} = x^3 - 3xy^2 + \dfrac{\partial C(y,z)}{\partial y} = x^3 - 3xy^2$

$\Rightarrow \; \dfrac{\partial C(y,z)}{\partial y} = 0 \; \Rightarrow \; C(y,z) = C(z) \Rightarrow U(x,y,z) = x^3y - xy^3 + C(z)$

$\hookrightarrow \; \dfrac{\partial U}{\partial z} = \dfrac{dC(z)}{dz} = 0 \; \Rightarrow \; C(z) = C$

$$\boxed{U(x,y,z) = x^3y - xy^3 + C}$$

3p

$$G: \left.\begin{aligned} \frac{x^2}{4} + \frac{y^2}{2} &= 2 \\ z &= 2 \end{aligned}\right\} \Rightarrow G \text{ egy ellipszis (zárt görbe)}$$

$\hookrightarrow \; \oint_G \underline{v}(\underline{r})\,d\underline{r} = 0$

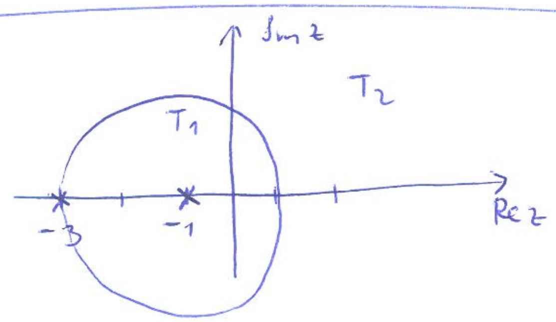
mert $\underline{v}(\underline{r})$ potenciálos

3p

(4) $f(z) = \dfrac{1}{(z+1)(z+3)}$ Laurent-sora $z_0 = -1$ körül

szinguláris pontok: $z = -1$

$z = -3$



lehetséges konvergencia gyűrűk:

- T_1: $0 < |z+1| < 2$
- T_2: $|z+1| > 2$

2p

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} = \frac{A(z+3)+B(z+1)}{(z+1)(z+3)} \rightsquigarrow A = \frac{1}{2}, \quad B = -\frac{1}{2}$$

$$\boxed{\frac{1}{(z+1)(z+3)} = \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right)}$$

2p

T_1-ben:

$$\frac{1}{z+3} = \frac{1}{2+(z+1)} = \frac{1}{2}\,\frac{1}{1+\frac{z+1}{2}} \stackrel{|\frac{z+1}{2}|<1}{=} \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z+1}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z+1)^n$$

$$\Rightarrow \boxed{f(z) = \frac{1}{2}\,\frac{1}{z+1} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(z+1)^n}$$

3p

T_2-ben:

$$\frac{1}{z+3} = \frac{1}{(z+1)+2} = \frac{1}{z+1}\,\frac{1}{1+\frac{2}{z+1}} \stackrel{|\frac{2}{z+1}|<1}{=} \frac{1}{z+1}\sum_{n=0}^{\infty}\left(-\frac{2}{z+1}\right)^n = \sum_{n=0}^{\infty}(-1)^n 2^n\left(\frac{1}{z+1}\right)^{n+1}$$

$$\Rightarrow f(z) = \frac{1}{2}\,\frac{1}{z+1} - \sum_{n=0}^{\infty}(-1)^n 2^{n-1}\left(\frac{1}{z+1}\right)^{n+1} = \left(\frac{1}{2}-\frac{1}{2}\right)\frac{1}{z+1} + \sum_{n=1}^{\infty}(-1)^{n+1}2^{n-1}\left(\frac{1}{z+1}\right)^{n+1}$$

$$\boxed{= \sum_{n=2}^{\infty}(-1)^n 2^{n-2}\left(\frac{1}{z+1}\right)^n = \frac{1}{(z+1)^2} - \frac{2}{(z+1)^3} + \frac{4}{(z+1)^4} - \dots}$$

3p

(5)

$$\underline{v}(\underline{r}) = \underline{r}\,\ln|\underline{r}| = \frac{1}{2}\ln(x^2+y^2+z^2)\cdot(x,y,z)$$

$\underline{r}\neq 0$

$\underline{r}=(x,y,z)$

$|\underline{r}|=\sqrt{x^2+y^2+z^2}$

$\boxed{1p}$

$$\frac{\partial v_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}\ln(x^2+y^2+z^2)\cdot x\right) = \frac{1}{2}\,\frac{2x}{x^2+y^2+z^2}\,x + \frac{1}{2}\ln(x^2+y^2+z^2) =$$

$$= \frac{x^2}{x^2+y^2+z^2} + \ln|\underline{r}|$$

hasonlóan: $\dfrac{\partial v_2}{\partial y} = \dfrac{y^2}{x^2+y^2+z^2} + \ln|\underline{r}|$, $\dfrac{\partial v_3}{\partial z} = \dfrac{z^2}{x^2+y^2+z^2} + \ln|\underline{r}|$

$$\Longrightarrow \boxed{\operatorname{div}\underline{v}} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} + 3\ln|\underline{r}| =$$

$$= \boxed{1 + 3\ln|\underline{r}|} \qquad \boxed{3p}$$

forrásmentesség: $1+3\ln|\underline{r}|=0 \iff |\underline{r}| = e^{-1/3}$

$\Updownarrow$

$\boxed{x^2+y^2+z^2 = e^{-2/3}}$ gömbön forrásmentes

$\boxed{2p}$

$$\operatorname{rot}\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$= \underline{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \underline{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) + \underline{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

pl: $\dfrac{\partial v_3}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{1}{2}\ln(x^2+y^2+z^2)\cdot z\right) = \dfrac{1}{2}\,\dfrac{2y}{x^2+y^2+z^2}\cdot z$

$\dfrac{\partial v_2}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{1}{2}\ln(x^2+y^2+z^2)\,y\right) = \dfrac{1}{2}\,\dfrac{2z}{x^2+y^2+z^2}\cdot y$

kiejtik egymást

$\boxed{1p}$

hasonlóan a többi is $\Longrightarrow$ $\boxed{\operatorname{rot}\underline{v}=0}$ $\boxed{3p}$ mindenhol örvénymentes