
1. **(25 pont)**
Mutassa meg, hogy az $y_1(x) = x$ és az $y_2(x) = x^2 - 1$ függvények az

$$y''(x) - \frac{2x}{x^2+1}y'(x) + \frac{2}{x^2+1}y(x) = 0$$

homogén lineáris differenciálegyenlet alapmegoldásait alkotják. Ezek segítségével adja meg az

$$y''(x) - \frac{2x}{x^2+1}y'(x) + \frac{2}{x^2+1}y(x) = x^3 + x$$

differenciálegyenlet általános megoldását!

2. **(25 pont)**
Melyek az $f(z) = \frac{z}{(z+1)(z+2)}$ függvény izolált szinguláris helyei! Adja meg az izol[olt]
szinguláris helyek körüli összes Laurent-sorát és azok konvergenciatartományát!

Melyek az $f(z) = \frac{\text{[illegible]}}{(z+1)(z+2)}$ függvény izolált szinguláris helyei! Adja meg az izolált szinguláris helyek körüli összes Laurent-sorát és azok konvergenciatartományát!

3. **(25 pont)**
Adja meg a $\mathbf{v}(x, y, z) = xy\,\mathbf{i} + y\,\mathbf{j} + x^2\,\mathbf{k}$ vektormező felületi integrálját a

$$z^2 = x^2 + y^2, \quad 0 \le z \le 1$$

képletekkel adott F felület mentén, ahol a felület normálisa a $\mathbf{k}$ egységvektorral tompaszöget zár be.

4. **(25 pont)**
Számolja ki a $\mathbf{v}(x, y, z) = yz(2x + y + z)\,\mathbf{i} + xz(x + 2y + z)\,\mathbf{j} + xy(x + y + 2z)\,\underline{k}$ erőtér divergenciáját és rotációját! Adja meg az erőtér munkáját az $O(0,0,0)$ pontból induló, $A(1,1,1)$ pontba érkező egyenesszakasz mentén!

1/

A3 1. vizsga – megoldások

(1) $y'' - \frac{2x}{x^2+1} y' + \frac{2}{x^2+1} y = 0$

- $y_1(x) = x \rightsquigarrow y_1'(x) = 1 \rightsquigarrow y_1''(x) = 0$

$\Rightarrow \quad 0 - \frac{2x}{x^2+1} \cdot 1 - \frac{2}{x^2+1} x = 0 \quad \Rightarrow y_1(x)$ mo $\boxed{2p}$

- $y_2(x) = x^2 - 1 \rightsquigarrow y_2'(x) = 2x \rightsquigarrow y_2''(x) = 2$

$\Rightarrow \quad 2 - \frac{2x}{x^2+1} 2x + \frac{2}{x^2+1}(x^2-1) = \frac{2x^2+2-4x^2+2x^2-2}{x^2+1} = 0 \Rightarrow y_2(x)$ mo $\boxed{3p}$

- függetlenség: Wronski-det:

$$W(x) = \det \begin{pmatrix} x & x^2-1 \\ 1 & 2x \end{pmatrix} = 2x^2 - (x^2-1) = x^2+1 \neq 0$$

$\Rightarrow$ y_1 és y_2 függetlenek $\Rightarrow$ alapmegoldásai a homogénnek: $\boxed{5p}$

$$\boxed{y_{H.ált}(x) = C_1 x + C_2 (x^2-1)}$$

IH: $y'' - \frac{2x}{x^2+1} y' + \frac{2}{x^2+1} y = x^3 + x$

$y_p(x) := C_1(x) \cdot x + C_2(x)(x^2-1)$ $\boxed{2p}$

$\hookrightarrow y_p'(x) = C_1'(x) \cdot x + C_1(x) + C_2'(x)(x^2-1) + C_2(x) \cdot 2x = C_1(x) + C_2(x) \cdot 2x$

tfh: $\boxed{C_1'(x) \cdot x + C_2'(x) \cdot (x^2-1) = 0}$ $(*)$ $\boxed{2p}$

$\hookrightarrow y_p''(x) = C_1'(x) + C_2'(x) \cdot 2x + 2 \cdot C_2(x)$

behelyettesítve: $C_1'(x) + C_2'(x) \cdot 2x + 2C_2(x) - \frac{2x}{x^2+1}(C_1(x) + C_2(x) \cdot 2x) +$

$+ \frac{2}{x^2+1}(C_1(x) \cdot x + C_2(x) \cdot (x^2-1)) = x^3 + x$

c/

① folyt. $\hookrightarrow$ $\boxed{C_1'(x) + 2x C_2'(x) = x^3+x}$ $(**)$ $\boxed{2p}$

$(*) + (**)$:

$$\begin{pmatrix} x & x^2-1 \\ 1 & 2x \end{pmatrix} \cdot \begin{pmatrix} C_1'(x) \\ C_2'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ x^3+x \end{pmatrix}$$

Cramer-szabály:

$$C_1'(x) = \frac{\det\begin{pmatrix} 0 & x^2-1 \\ x^3+x & 2x \end{pmatrix}}{W(x)} = \frac{-(x^2-1)\overbrace{(x^3+x)}^{x(x^2+1)}}{x^2+1} = x - x^3$$

$\hookrightarrow$ $C_1(x) = \int (x-x^3)\,dx = \frac{x^2}{2} - \frac{x^3}{3}$ $\boxed{3p}$

$$C_2'(x) = \frac{\det\begin{pmatrix} x & 0 \\ 1 & x^3+x \end{pmatrix}}{W(x)} = \frac{x(x^3+x)}{x^2+1} = x^2$$

$\hookrightarrow$ $C_2(x) = \int x^2\,dx = \frac{x^3}{3}$ $\boxed{3p}$

$$\Rightarrow y_{ip}(x) = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)x + \frac{x^3}{3}(x^2-1) = \frac{1}{6}\left(3x^3 - 2x^4 + 2x^5 - 2x^3\right) =$$

$$= \frac{2x^5 - 2x^4 - x^3}{6}$$

$$\boxed{y_{ált}(x) = C_1 x + C_2(x^2-1) + \frac{2x^5-2x^4-x^3}{6}}$$

$\boxed{3p}$

3/

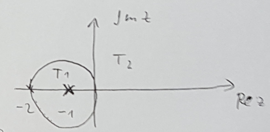
(2) $$f(z) = \frac{z}{(z+1)(z+2)}$$ $\leftarrow$ holomorf $\Rightarrow$ szing. helyek: $z=-1,\ z=-2$ [2p]

$$f(z) = \frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} = \frac{A(z+2)+B(z+1)}{(z+1)(z+2)}$$

számlálóban:
- $z=-2 \rightsquigarrow -B=-2 \Rightarrow B=2$
- $z=-1 \rightsquigarrow A=-1$

$\hookrightarrow$ $$f(z) = \frac{2}{z+2} - \frac{1}{z+1}$$ [3p]

a) $z=-1$ körüli sorfejtés



T_1: $0<|z+1|<1$

T_2: $|z+1|>1$ [2p]

T_1-n: $$\frac{2}{z+2} = \frac{2}{1+(z+1)} \underset{|z+1|<1}{=} 2\sum_{n=0}^{\infty}(-1)^n(z+1)^n$$

$\hookrightarrow$ $$\boxed{f(z) = -\frac{1}{z+1} + 2\sum_{n=0}^{\infty}(-1)^n(z+1)^n}$$ [4p]

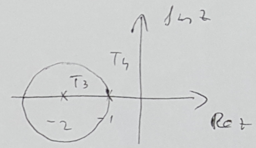
- T_2-n: $$\frac{2}{z+2} = \frac{2}{(z+1)+1} = \frac{2}{z+1}\cdot\frac{1}{1+\frac{1}{z+1}} \underset{\left|\frac{1}{z+1}\right|<1}{=} \frac{2}{z+1}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z+1}\right)^n$$

$\hookrightarrow$ $$\boxed{f(z) = -\frac{1}{z+1} + \frac{2}{z+1}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(z+1)^n}} =$$ [4p]

$$= \frac{1}{z+1} + \sum_{n=1}^{\infty} 2(-1)^n \frac{1}{(z+1)^{n+1}}$$

② folyt

b) $z = -2$ körüli sorfejtés



T_3: $0 < |z+2| < 1$

T_4: $1 < |z+2|$

$\boxed{2p}$

- T_3-n

$$-\frac{1}{z+1} = -\frac{1}{(z+2)-1} = \frac{1}{1-(z+2)} \underset{|z+2|<1}{=} \sum_{n=0}^{\infty} (z+2)^n$$

$\hookrightarrow \boxed{f(z) = \frac{2}{z+2} + \sum_{n=0}^{\infty} (z+2)^n}$ $\boxed{4p}$

- T_4-n

$$-\frac{1}{z+1} = -\frac{1}{(z+2)-1} = -\frac{1}{z+2} \cdot \frac{1}{1-\frac{1}{z+2}} \underset{\left|\frac{1}{z+2}\right|<1}{=}$$

$$= -\frac{1}{z+2} \sum_{n=0}^{\infty} \left(\frac{1}{z+2}\right)^n$$

$\hookrightarrow \boxed{f(z) = \frac{2}{z+2} - \frac{1}{z+2} \sum_{n=0}^{\infty} \left(\frac{1}{z+2}\right)^n} = \frac{1}{z+2} - \sum_{n=1}^{\infty} \left(\frac{1}{z+2}\right)^{n+1}$

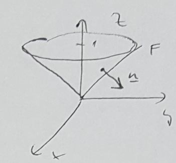
$\boxed{4p}$

5/

(3) $\underline{v}(x,y,z) = xy\,\underline{i} + y\,\underline{j} + x^2\,\underline{k}$

$F:\quad z^2 = x^2 + y^2$

$\quad\;\; 0 \le z \le 1$



kúp palást

paraméteres:

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = u \end{cases} \qquad \left.\begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 2\pi \end{matrix}\right\} T \qquad \boxed{3p}$$

$$\underline{r}(u,v) = u\cos v\,\underline{i} + u\sin v\,\underline{j} + u\,\underline{k}$$

↳ $\underline{r}'_u(u,v) = \cos v\,\underline{i} + \sin v\,\underline{j} + \underline{k}$ $\boxed{2p}$

$\underline{r}'_v(u,v) = -u\sin v\,\underline{i} + u\cos v\,\underline{j}$ $\boxed{2p}$

$$\underline{r}'_u \times \underline{r}'_v = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{pmatrix} = \underline{i}(-u\cos v) - \underline{j}(+u\sin v) + \underline{k}\,u = (-u\cos v, -u\sin v, u) \quad \boxed{3p}$$

$(\underline{r}'_u \times \underline{r}'_v)\cdot\underline{k} = u \ge 0 \rightsquigarrow$ $\underline{r}'_u \times \underline{r}'_v$ hegyesszöget zár be $\underline{k}$-val

↳ $-(\underline{r}'_u \times \underline{r}'_v)$-t kell tekinteni $\boxed{2p}$

$$\underline{v}(\underline{r}(u,v)) = u^2\cos v\sin v\,\underline{i} + u\sin v\,\underline{j} + u^2\cos^2 v\,\underline{k}$$

$$\underline{v}(\underline{r}(u,v))\cdot(\underline{r}'_u \times \underline{r}'_v) = -u^3\cos^2 v\sin v - u^2\sin^2 v + u^3\cos^2 v$$

$$\iint_F \underline{v}(\underline{r})\,d\underline{F} = -\iint_T \underline{v}(\underline{r}(u,v))\cdot(\underline{r}'_u \times \underline{r}'_v)\,du\,dv =$$

$$= -\int_0^1\int_0^{2\pi}\left(-u^3\cos^2 v\sin v - u^2\sin^2 v + u^3\cos^2 v\right)dv\,du \;=$$

$\boxed{5p}$

c/ (3) pót

$$= -\int_0^1 \left[u^3 \frac{\cos^3 v}{3} - \frac{u^2}{2}\left(v - \frac{\sin 2v}{2}\right) + \frac{u^3}{2}\left(v + \frac{\sin 2v}{2}\right)\right]_0^{2\pi} du =$$

$\boxed{4p}$

$$\left.\begin{array}{l} \cos^2 v + \sin^2 v = 1 \\ \cos^2 v - \sin^2 v = \cos 2v \end{array}\right\} \Rightarrow \begin{array}{l} \sin^2 v = \frac{1-\cos 2v}{2} \\ \cos^2 v = \frac{1+\cos 2v}{2} \end{array}$$

$$= -\int_0^1 \left(-u^2\pi + u^3\pi\right) du = -\pi\left[-\frac{u^3}{3} + \frac{u^4}{4}\right]_0^1 = -\pi\left(-\frac{1}{3} + \frac{1}{4}\right) = \frac{\pi}{12}$$

$\boxed{4p}$

2. mo. felület leírás + Gauss–Osztrogradszkij

B/

(5) $\underline{v}(x,y,z) = yz(2x+y+z)\underline{i} + xz(x+2y+z)\underline{j} + xy(x+y+2z)\underline{k}$

$$\operatorname{div}\underline{v} = 2yz + 2xz + 2xy \qquad \boxed{1p}$$

$$\operatorname{rot}\underline{v} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz(2x+y+z) & xz(x+2y+z) & xy(x+y+2z) \end{pmatrix} =$$

$$= \underline{i}\,\big(\overbrace{x(x+y+2z)+xy - x(x+2y+z)}^{=0} - xz\big)$$

$$- \underline{j}\,\big(\overbrace{y(x+y+2z)+xy - y(2x+y+z)}^{=0} - yz\big)$$

$$+ \underline{k}\,\big(\overbrace{z(x+2y+z)+xz - z(2x+y+z)}^{=0} - yz\big) = \underline{0} \qquad \boxed{1p}$$

$\operatorname{rot}\underline{v} = \underline{0} \Rightarrow \exists\ U(\underline{r})$ potenciál, hogy $\underline{v} = \operatorname{grad} U$

azaz

$$\begin{cases} \dfrac{\partial U}{\partial x} = yz(2x+y+z) \\ \dfrac{\partial U}{\partial y} = xz(x+2y+z) \\ \dfrac{\partial U}{\partial z} = xy(x+y+2z) \end{cases}$$

$\hookrightarrow\ U = \int \big(yz(2x+y+z)\big)\,dx = x^2yz + xyz(y+z) + C(y,z)$

$\leadsto\ \dfrac{\partial U}{\partial y} = x^2z + xz(y+z) + xyz + \dfrac{\partial C(y,z)}{\partial y} = xz(x+2y+z)$

$\hookrightarrow\ \dfrac{\partial C(y,z)}{\partial y} = 0 \Rightarrow C(y,z) = C(z)$

8/ $\Rightarrow \quad u = x^2yz + xy^2z + xyz^2 + C(z)$

$\hookrightarrow \quad \dfrac{\partial u}{\partial z} = x^2y + xy^2 + 2xyz + \dfrac{dC(z)}{dz} = x^2y + xy^2 + 2xyz$

$\hookrightarrow \quad \dfrac{dC(z)}{dz} = 0 \Rightarrow C(z) = c$

a potenciálfv.:

$$\boxed{u(x,y,z) = x^2yz + xy^2z + xyz^2 + c = xyz(x+y+z) + c}$$

$\boxed{10p}$

$$\int_G \underline{v}(\underline{r})\,d\underline{r} = u(1,1,1) - u(0,0,0) = 3 - 0 = \underline{\underline{3}}$$

$\boxed{5p}$

G: $O(0,0,0) \to A(1,1,1)$
szakasz

2. mo görbementi integrál

G: $\underline{r}(t) = t(1,1,1) \quad 0 \le t \le 1$

$\dot{\underline{r}}(t) = (1,1,1)$

$\underline{v}(\underline{r}(t)) = (4t^3, 4t^3, 4t^3)$

$\boxed{10p}$

$$\int_G \underline{v}(\underline{r})\,d\underline{r} = \int_0^1 12t^3\,dt = \left[\frac{12t^4}{4}\right]_0^1 = \frac{12}{4} = \underline{\underline{3}}$$

$\boxed{5p}$