① $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ egyenletesen konv $[-a,a]$-n $\forall a \in \mathbb{R}$ esetén

↳ $e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}$ szintén egy. konv $[-a,a]$-n $\forall a \in \mathbb{R}$-re

$\boxed{2 \text{ pont}}$

$$I = \int_{-0,1}^{0,1} e^{-x^2}\,dx \underset{\uparrow}{=} 2\int_0^{0,1} e^{-x^2}\,dx = 2\int_0^{0,1}\left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}\right)dx \underset{\uparrow}{=} 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_0^{0,1} x^{2k}\,dx$$

páros fv   egy. konv.

$$= 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left[\frac{x^{2k+1}}{2k+1}\right]_0^{0,1} = 2\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(2k+1)}\left(\frac{1}{10}\right)^{2k+1}$$

$\boxed{3 \text{ pont}}$

↑ Leibniz-sor

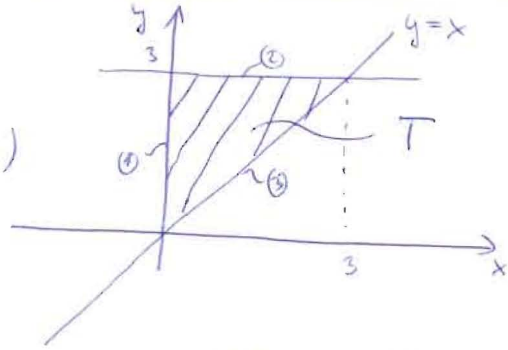
$S_N = \sum_{k=0}^{N-1} \frac{2(-1)^k}{k!(2k+1)}\left(\frac{1}{10}\right)^{2k+1}$ -nyel való becslés elkövetett hiba: $\boxed{3 \text{ pont}}$

$$H = |I - S_N| < \frac{2}{N!(2N+1)}\left(\frac{1}{10}\right)^{2N+1} < 10^{-4}$$ biztos, ha pl $N=3$ $\boxed{2 \text{ pont}}$

↳ $$I \approx \sum_{k=0}^{2} \frac{2(-1)^k}{k!(2k+1)}\left(\frac{1}{10}\right)^{2k+1} = 2\left(\frac{1}{10} - \frac{1}{3}\frac{1}{1000} + \frac{1}{2\cdot 5}\frac{1}{100000}\right)$$

---

② $f(x,y) = 2x^2 + y^2 - 4x - 4y$



$f'_x(x,y) = 4x - 4 = 0$ (szükséges feltétel)

$f'_y(x,y) = 2y - 4 = 0$

⇕

P: $x=1,\ y=2$ $P(1,2) \in T$ Hesse-mátrix

$f''_{xx}(x,y) = 4$ $f''_{xy}(x,y) = f''_{yx}(x,y) = 0$ $f''_{yy}(x,y) = 2$ $H = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

$\det H = 8 > 0 \leadsto \exists$ lok. n.é. P-ben és ez MINIMUM, mert $H_{11} = 4 > 0$

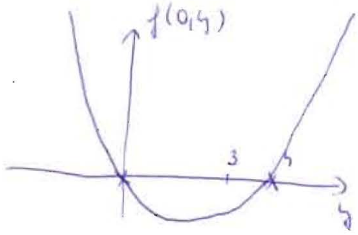
$\Rightarrow$ $\boxed{f(1,2) = -6 \text{ lok. MIN}}$ $\boxed{3 \text{ pont}}$

határok: ① $x = 0,\ 0 \le y \le 3$

$f(0,y) = y^2 - 4y = y(y-4)$

min, ha $y=2$ $f(0,2) = -4$

max, ha $y=0$ $f(0,0) = 0$



$\boxed{2 \text{ pont}}$

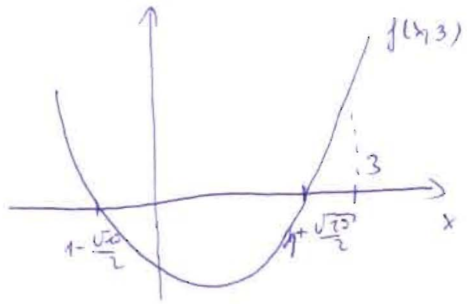
② folytatás

② $\underline{y=3,\ 0\le x\le 3}$ $\rightsquigarrow$ $f(x,3)=2x^2-4x-3$ gyökei: $1\pm\frac{\sqrt{10}}{2}$

MIN: $\dfrac{1+\frac{\sqrt{10}}{2}+1-\frac{\sqrt{10}}{2}}{2}=1$-nél

$f(1,3)=-5$

MAX: $f(3,3)=3$



||2 pont

③ $\underline{y=x\quad 0\le x\le 3}$

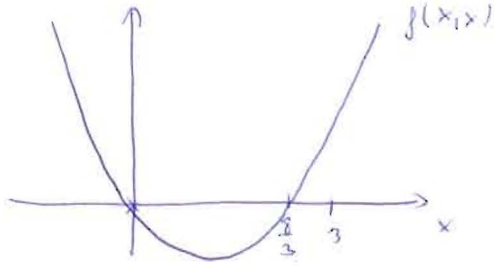
$f(x,x)=3x^2-8x=x(3x-8)$ gyökei: $x=0$, $x=\frac{8}{3}$

MIN: $x=\frac{4}{3}$-nál

$f\left(\frac{4}{3},\frac{4}{3}\right)=-\frac{16}{3}$

MAX $x=3$-nál

$f(3,3)=3$



||2 pont

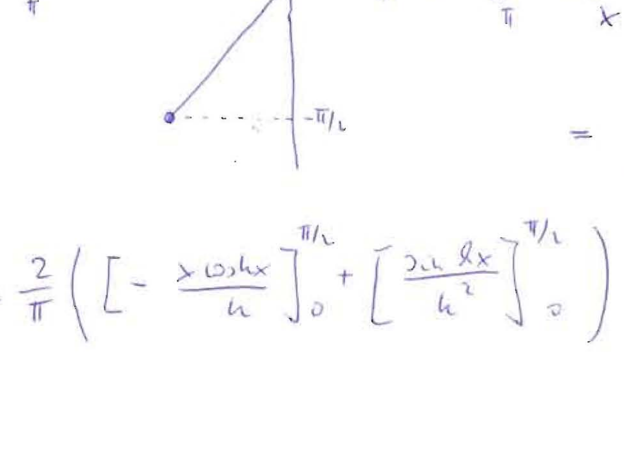
Összefoglalva: abszolút MIN : $P(1,2)$-nél $f(1,2)=-6$ lokális is

abszolút MAX $P(3,3)$-nál $f(3,3)=3$

||1 pont

---

③



f 1 periódusa:

||1 pont

páratlan $\Longrightarrow$ $a_k=0$ $\forall k=0,1,2,\dots$

$$b_k=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx=$$

$$=\frac{2}{\pi}\int_0^{\pi/2} x\sin kx\,dx \underset{\text{parc. int}}{=} \frac{2}{\pi}\left\{\left[-x\frac{\cos kx}{k}\right]_0^{\pi/2}+\int_0^{\pi/2}\frac{\cos kx}{k}\,dx\right\}$$

$$=\frac{2}{\pi}\left(\left[-\frac{x\cos kx}{k}\right]_0^{\pi/2}+\left[\frac{\sin kx}{k^2}\right]_0^{\pi/2}\right)=\frac{2}{\pi}\cdot\frac{\sin k\frac{\pi}{2}}{k^2}-\frac{\cos\left(k\frac{\pi}{2}\right)}{k}$$

||4 pont

③ folyt

$$\sin k\cdot\frac{\pi}{2} = \begin{cases} 0 & k \text{ páros} \\ (-1)^l, & \text{ha } k=2l+1 \end{cases}$$

$$\cos k\cdot\frac{\pi}{2} = \begin{cases} 0 & k \text{ páratlan} \\ (-1)^l & \text{ha } k=2l \end{cases}$$

$\boxed{2 \text{ pont}}$

$$\Rightarrow \quad b_{2l} = \frac{(-1)^{l+1}}{2l} \qquad b_{2l+1} = \frac{2}{\pi}\frac{(-1)^l}{(2l+1)^2}$$

$\boxed{1 \text{ pont}}$

$$\Phi(x) = \sum_{l=0}^{\infty} \frac{2}{\pi}\frac{(-1)^l}{(2l+1)^2}\sin(2l+1)x + \sum_{l=1}^{\infty}\frac{(-1)^{l+1}}{2l}\sin 2lx =$$

$$= \frac{2}{\pi}\sin x + \frac{1}{2}\sin 2x - \frac{2}{\pi}\frac{1}{3^2}\sin 3x - \frac{1}{4}\sin 4x + \dots$$

A Dirichlet-tétel értelmében a szakadási pontok kivételével előállítja a Fourier-sor $f(x)$-et, azaz valahányszor $x \neq \pm\frac{\pi}{2} + 2k\pi$

$\boxed{2 \text{ pont}}$

---

⑤ $f(x,y) = \sqrt{2x^2+y^2}$

a)
$$\left.\begin{aligned} f'_x(x,y) &= \frac{2x}{\sqrt{2x^2+y^2}} \\ f'_y(x,y) &= \frac{y}{\sqrt{2x^2+y^2}} \end{aligned}\right\} \text{ folytonosak, ha } (x,y)\neq(0,0)$$

$\Downarrow$

$$\exists \operatorname{grad} f = \left(\frac{2x}{\sqrt{2x^2+y^2}}, \frac{y}{\sqrt{2x^2+y^2}}\right)$$

$$f'_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{2h^2}-0}{h} = \cancel{\sqrt{2}\,|h|}\lim_{h\to 0}\frac{\sqrt{2}\,|h|}{h} \quad \nexists$$

$\hookrightarrow$ $(x,y)=(0,0)$-ban nem deriválható

$\boxed{4 \text{ pont}}$

b) $P(2,1)$ $\quad z_0 = f(2,1) = 3 \quad f'_x(2,1) = \frac{4}{3} \quad f'_y(2,1) = \frac{1}{3}$

$\hookrightarrow$ érintősík: $\boxed{z-3 = \frac{4}{3}(x-2) + \frac{1}{3}(y-1)}$

$\boxed{3 \text{ pont}}$

c) $\underline{v} = (3,4) \rightsquigarrow |\underline{v}| = \sqrt{9+16} = 5 \rightsquigarrow \underline{v}^0 = \frac{1}{5}(3,4) \qquad \operatorname{grad} f|_P = \left(\frac{4}{3}, \frac{1}{3}\right)$

$$\left.\frac{df}{d\underline{e}}\right|_{(2,1)} = \langle \operatorname{grad} f|_P, \underline{e}\rangle = \frac{3}{5}\cdot\frac{4}{3} + \frac{4}{5}\cdot\frac{1}{3} = \boxed{\frac{16}{15}}$$

$\boxed{3 \text{ pont}}$

(5) $p(x) = x^3 - 2x^2 + 2x - 6 = a(x^3 + \alpha x) + b(3x^3 + x^2) + c(x^2 + 2x + 4)$

$\Updownarrow$ $\forall x \in \mathbb{R}$

$$\left.\begin{array}{lllll} x^3: & a + 3b & & & = 1 \\ x^2: & & b + c & & = -2 \\ x: & \alpha a & & +2c & = 2 \\ \text{konst}: & & & 4c & = -6 \end{array}\right\} \Rightarrow c = -\frac{3}{2}$$

4 pont

$\Downarrow$ 2. egyenlet

$b = -2 - c = -2 + \frac{3}{2} = -\frac{1}{2}$

$\Leftarrow$ 1. egyenlet

$a = 1 - 3b = 1 + \frac{3}{2} = \frac{5}{2}$

$\Leftarrow$ 3. egyenlet

$\alpha \cdot \frac{5}{2} - 3 = 2 \Rightarrow \boxed{\alpha = 2}$

6 pont

$\alpha = 2$ esetén

$$\boxed{p = \frac{5}{2} q_1 - \frac{1}{2} q_2 - \frac{3}{2} q_3}$$