1. Vizsgadolgozat – javítási útmutató

① $\underbrace{(1-xy)}_{P(x,y)}\,dx + \underbrace{(xy-x^2)}_{Q(x,y)}\,dy = 0 \qquad y(1)=4$

$$\frac{\partial P}{\partial y} = -x \neq \frac{\partial Q}{\partial x} = y-2x \rightsquigarrow \text{nem egzakt}$$

Tfh, $\exists\ \mu=\mu(x)$ integráló tényező. Ezzel beszorozva:

$$\underbrace{(1-xy)\mu(x)}_{\tilde{P}(x,y)}\,dx + \underbrace{(xy-x^2)\mu(x)}_{\tilde{Q}(x,y)}\,dy = 0$$

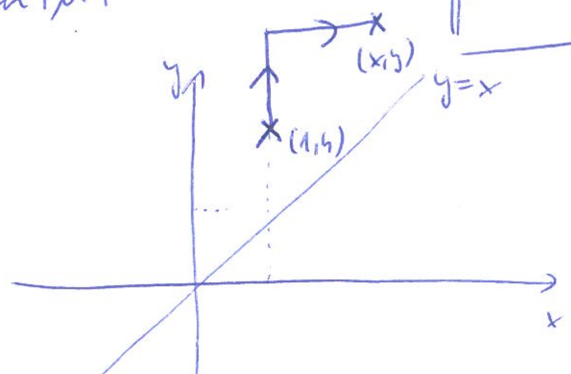
Egzaktság feltétele

$$\frac{\partial \tilde{P}}{\partial y} = -x\mu(x) = \frac{\partial \tilde{Q}}{\partial x} = (y-2x)\mu(x) + (xy-x^2)\mu'(x)$$

rendezve: $(x-y)\mu(x) = x(y-x)\mu'(x)$

$x \neq y \quad \Rightarrow \quad \dfrac{d\mu}{\mu} = -\dfrac{1}{x}\,dx$

$\ln|\mu| = -\ln|x| \quad \Rightarrow \quad \boxed{\mu(x) = \frac{1}{x}}$



5

↳ $\tilde{P}(x,y) = \dfrac{1}{x} - y$

$\tilde{Q}(x,y) = y - x$

$$\boxed{F(x,y)} = \int_4^y \tilde{Q}(1,t)\,dt + \int_1^x \tilde{P}(t,y)\,dt =$$

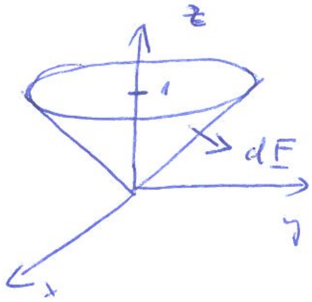
5

$$= \int_4^y (t-1)\,dt + \int_1^x \left(\frac{1}{t} - y\right)dt = \left[\frac{t^2}{2} - t\right]_4^y + \left[\ln|t| - yt\right]_1^x =$$

$$= \frac{y^2}{2} - y - 8 + 4 + \ln|x| - yx - \ln 1 + y = \boxed{\frac{y^2}{2} - yx + \ln|x| - 4 = 0}$$

② $\underline{v}(x,y,t) = xy\,\underline{i} + y\,\underline{j} + x^2\,\underline{k}$

$F: \left.\begin{array}{l} z^2 = x^2+y^2 \\ 0 \le z \le 1 \end{array}\right\}$



forgáskúp

$\iint\limits_F \underline{v}(\underline{r})\,d\underline{F} = ?$

__1. mo:__ felszíni integrállal – forgáskúp paraméterezése:

$x^2+y^2=z^2$ ✓ $\quad \left\{\begin{array}{l} x = u\cos v \\ y = u \sin v \\ z = u \end{array}\right.$

$\boxed{\begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le 2\pi \end{array}}$

$\boxed{3}$

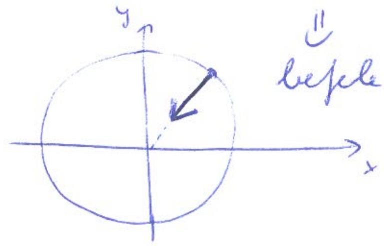
$\underline{r}(u,v) = u\cos v\,\underline{i} + u\sin v\,\underline{j} + u\cdot\underline{k}$

$\left.\begin{array}{l} \underline{r}'_u = \cos v\,\underline{i} + \sin v\,\underline{j} + \underline{k} \\ \underline{r}'_v = -u\sin v\,\underline{i} + u\cos v\,\underline{j} \end{array}\right\} \Rightarrow \quad \underline{r}'_u \times \underline{r}'_v =$

$= \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{pmatrix} =$

$= -u\cos v\,\underline{i} - u\sin v\,\underline{j} + u\,\underline{k}$

$\underline{v}(\underline{r}(u,v)) = u^2\cos v\sin v\,\underline{i} + u\sin v\,\underline{j} +$
$\quad + u^2\cos^2 v\,\underline{k}$



⇑ befelé áll

$\boxed{3}$

$-\underline{v}(\underline{r}(u,v))\cdot(\underline{r}'_u \times \underline{r}'_v) =$

$= u^3\cos^2 v\sin v + u^2\sin^2 v - u^3\cos^2 v$

$\iint\limits_F \underline{v}(\underline{r})\,d\underline{F} = \int_0^{2\pi}\int_0^1 (u^3\cos^2 v\sin v + u^2\sin^2 v - u^3\cos^2 v)\,du\,dv =$

$= \int_0^{2\pi} \left[\frac{u^4}{4}\cos^2 v(\sin v - 1) + \frac{u^3}{3}\sin^2 v\right]_0^1 = \int_0^{2\pi}\left(\frac{1}{4}\cos^2 v\sin v - \frac{1}{4}\cos^2 v + \frac{1}{3}\sin^2 v\right)dv$

$= \left[-\frac{1}{4}\,\frac{\cos^3 v}{3} - \frac{1}{8}\left(v + \frac{\sin 2v}{2}\right) + \frac{1}{6}\left(v - \frac{\sin 2v}{2}\right)\right]_0^{2\pi} = -\frac{2\pi}{8} + \frac{2\pi}{6} = \underline{\underline{\frac{\pi}{12}}}$

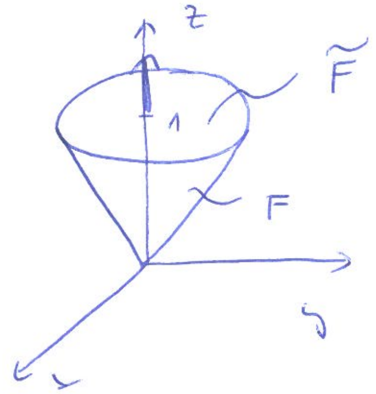
↑

$\cos^2 v = \frac{1+\cos 2v}{2} \quad | \quad \sin^2 v = \frac{1-\cos 2v}{2}$

$\boxed{4}$

(2)

2. mo



fedjük le az $\tilde{F}$ fedőlappal

⇓

$$\iint_F \underline{v}(\underline{r})\,d\underline{F} = \oiint_{F\cup\tilde{F}} \underline{v}(\underline{r})\,d\underline{F} - \iint_{\tilde{F}} \underline{v}\,d\underline{F}$$

$$\oiint_{F\cup\tilde{F}} \underline{v}(\underline{r})\,d\underline{F} \underset{\text{Gauss-Osztr.}}{=} \iiint_V \operatorname{div}\underline{v}\,dV =$$

3

$$\operatorname{div}\underline{v} = y+1$$

kúp hengerkoordinátákban:

$$\boxed{\begin{array}{l} 0\le r\le z \\ 0\le \varphi \le 2\pi \\ 0\le z\le 1 \end{array}} \quad \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$$

$$|J| = r$$

$$= \int_0^1\int_0^{2\pi}\underbrace{\int_0^z (r\sin\varphi+1)\,r\,dr}_{\left[\frac{r^3}{3}\sin\varphi+\frac{r^2}{2}\right]_0^z = \frac{z^3}{3}\sin\varphi+\frac{z^2}{2}}\,d\varphi\,dz = \int_0^1\left[-\frac{z^3}{3}\cos\varphi+\frac{z^2}{2}\varphi\right]_0^{2\pi} dz =$$

4

$$= \pi\int_0^1 z^2\,dz = \pi\left[\frac{z^3}{3}\right]_0^1 = \underline{\underline{\frac{\pi}{3}}}$$

$$\tilde{F}:\ \underline{r}(u,v) = u\cos v\,\underline{i} + u\sin v\,\underline{j} + \underline{k} \qquad \boxed{0\le u\le 1,\ 0\le v\le 2\pi}$$

$$\left.\begin{array}{l} \underline{r}'_u = \cos v\,\underline{i} + \sin v\,\underline{j} \\ \underline{r}'_v = -u\sin v\,\underline{i} + u\cos v\,\underline{j} \end{array}\right\} \Rightarrow \underline{r}'_u\times\underline{r}'_v = u\,\underline{k}$$

$$\underline{v}(\underline{r}(u,v)) = (u^2\cos v\sin v,\ u\sin v,\ u^2\cos^2 v)$$

$$\iint_{\tilde{F}} \underline{v}(\underline{r})\,d\underline{F} = \int_0^{2\pi}\underbrace{\int_0^1 u^3\cos^2 v\,du}_{\left[\frac{u^4}{4}\cos^2 v\right]_0^1 = \frac{\cos^2 v}{4}}\,dv = \frac{1}{8}\left[v+\frac{\sin 2v}{2}\right]_0^{2\pi} = \underline{\underline{\frac{\pi}{4}}}$$

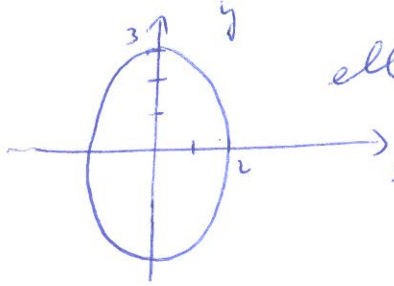
3

végeredmény: $\dfrac{\pi}{3}-\dfrac{\pi}{4} = \boxed{\underline{\underline{\dfrac{\pi}{12}}}}$

③ $\underline{v}(\underline{r}) = \underline{r} + \underline{k} \times \underline{r} \underset{\uparrow}{=} (x-y)\,\underline{i} + (x+y)\,\underline{j} + z\,\underline{k}$

$$\underline{k} \times \underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & 1 \\ x & y & z \end{vmatrix} = -y\,\underline{i} + x\,\underline{j}$$ $\boxed{3}$

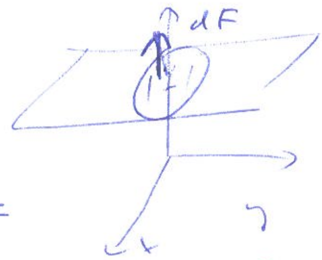
<u>1. mo</u> $\quad G: \ \frac{x^2}{4} + \frac{y^2}{9} = 1, \ z = 2$



ellipszis a $z=2$ síkon

$$\oint_G \underline{v}(\underline{r})\,d\underline{r} \underset{\uparrow}{=} \iint_F \text{rot}\,\underline{v}\,d\underline{F}$$

Stokes-tétel $\boxed{3}$



$$\text{rot}\,\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x-y & x+y & z \end{vmatrix} = 2\,\underline{k}$$ és $d\underline{F} \parallel \underline{k}$ $\boxed{4}$

$$\Rightarrow \iint_F \text{rot}\,\underline{v}\,d\underline{F} = 2 \iint_F |d\underline{F}| = 2 \cdot T_{\text{ellipszis}} = 2 \cdot \pi \cdot 2 \cdot 3 = \underline{\underline{12\pi}}$$

<u>2. mo</u> $\quad G$ ellipszis paraméterezése: $\underline{r}(t) = 2\cos t\,\underline{i} + 3\sin t\,\underline{j} + 2\,\underline{k}$

$0 \le t \le 2\pi \qquad \hookrightarrow \dot{\underline{r}}(t) = -2\sin t\,\underline{i} + 3\cos t\,\underline{j}$ $\boxed{3}$

$$\underline{v}(\underline{r}(t)) = (2\cos t - 3\sin t)\,\underline{i} + (2\cos t + 3\sin t)\,\underline{j} + 2\,\underline{k}$$

$$\oint_G \underline{v}(\underline{r})\,d\underline{r} = \int_0^{2\pi} \left(-4\cos t \sin t + 6\sin^2 t + 6\cos^2 t + 9\sin t \cos t\right) dt =$$

$$= \int_0^{2\pi} \left(5\cos t \sin t + 6\right) dt = \left[5\,\frac{\sin^2 t}{2} + 6t\right]_0^{2\pi} = \underline{\underline{12\pi}}$$

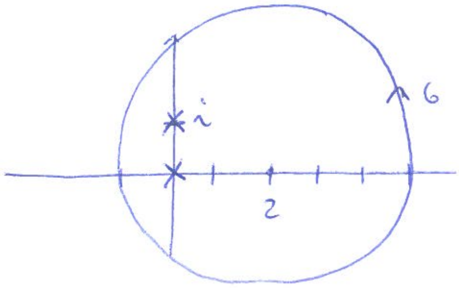
$\boxed{4}$

(4) $$\oint_{|z-2|=3} \left(\bar{z} + \frac{1}{z^3(z-i)} + z^3 e^{\frac{1}{z}} \cos\frac{1}{z} \right) dz$$

szingularitások: • $z=0$

• $z=i$

G belsejében vannak



részekre bontva:

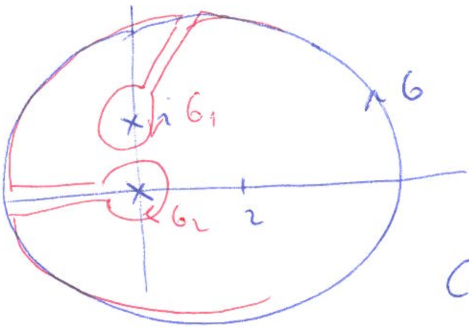
• $$\oint_{|z-2|=3} \bar{z}\, dz = \int_0^{2\pi} (2+3e^{-it})\cdot(3ie^{it})\,dt = 6i\underbrace{\int_0^{2\pi} e^{it}dt}_{\left[\frac{e^{it}}{i}\right]_0^{2\pi}=0} + 9i\int_0^{2\pi} dt = \underline{\underline{18\pi i}}$$

$z(t) = 2+3e^{it}$ $\quad 0 \le t \le 2\pi$

$\dot{z}(t) = 3ie^{it}$

3

• $$\oint_{|z-2|=3} \frac{1}{z^3(z-i)}\,dz = \oint_{G_1} \frac{\frac{1}{z^3}}{z-i}\,dz + \oint_{G_2} \frac{\frac{1}{z-i}}{z^3}\,dz =$$



$$= 2\pi i \left.\frac{1}{z^3}\right|_{z=i} + \frac{2\pi i}{2!}\left.\left(\frac{1}{z-i}\right)''\right|_{z=0} =$$

Cauchy-int.formula

$$= -2\pi + \pi i \left.\frac{2}{(z-i)^3}\right|_{z=0} = -2\pi + 2\pi = \underline{\underline{0}}$$

3

• $$z^3 e^{\frac{1}{z}} = z^3 \cdot \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = z^3\left(1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \dots\right) = z^3 + z^2 + \frac{1}{2}z + \frac{1}{3!} + \frac{1}{4!}\cdot\frac{1}{z} + \dots$$

$$\cos\frac{1}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{z}\right)^{2k} = 1 - \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{4!}\cdot\frac{1}{z^4} + \dots$$

$z^3 e^{\frac{1}{z}}\cdot\cos\frac{1}{z}$ 0 körüli Laurent-sorában $\frac{1}{z}$ együtthatója:

$$c_{-1} = \underset{z^3\cdot\frac{1}{z^4}\text{-ből}}{\frac{1}{4!}} - \underset{z\cdot\frac{1}{z^2}\text{-ből}}{\frac{1}{2}\cdot\frac{1}{2!}} + \underset{1\cdot\frac{1}{z}\text{-ből}}{\frac{1}{4!}} = \frac{2}{4!} - \frac{1}{4} = \frac{1}{12} - \frac{1}{4} = \underline{\underline{-\frac{1}{6}}}$$

(4) folytatás: $\boxed{3}$

$$\oint_{|z-2|=3} z^3 e^{\frac{1}{z}} \cos\frac{1}{z}\,dz \underset{\substack{\uparrow \\ \text{reziduum-tétel}}}{=} 2\pi i\, c_{-1} = -2\pi i \cdot \frac{1}{6} = -\frac{\pi i}{3}$$

<u>végeredmény</u> $= 18\pi i + 0 - \frac{\pi i}{3} = \frac{53\pi i}{3}$ $\boxed{1}$

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(5) a) $\underline{v}(\underline{r}) = (x^2-y^2)\,\underline{i} + (y^2-z^2)\,\underline{j} + (z^2-x^2)\,\underline{k}$

$$\operatorname{div} \underline{v} = 2x + 2y + 2z = 2(x+y+z)$$ $\boxed{1}$

$$\operatorname{rot} \underline{v} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2 & y^2-z^2 & z^2-x^2 \end{pmatrix} = \underline{i}\,(0-2z) - \underline{j}\,(-2x-0) + \underline{k}\,(0-2y) =$$
$$= -2\,(z, -x, y)$$ $\boxed{2}$

- forrásmentes az $x+y+z=0$ síkon $\boxed{1}$
- örvénymentes az origóban $\boxed{1}$

b) $\underline{v}(\underline{r}) = \frac{x}{y}\,\underline{i} + \frac{y}{z}\,\underline{j} + xz\,\underline{k}$

$$\operatorname{div} \underline{v} = \frac{1}{y} + \frac{1}{z} + x$$ $\boxed{1}$

$$\operatorname{rot} \underline{v} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{y} & \frac{y}{z} & xz \end{pmatrix} = \underline{i}\left(0-\frac{y}{z^2}\right) - \underline{j}\,(z-0) + \underline{k}\left(0-\frac{x}{y^2}\right) =$$
$$= \left(-\frac{y}{z^2}, -z, -\frac{x}{y^2}\right)$$ $\boxed{2}$

- forrásmentes az $\frac{1}{y}+\frac{1}{z}+x=0$ felületen $\boxed{1}$
- sehol sem örvénymentes $\boxed{1}$