MINTAMEGOLDÁS — 1 — (anal 2, 1. ZH, β, 2013. 03. 14.)

1, $\boxed{13}$ $y' = \dfrac{\operatorname{tg} y}{\sqrt{x^2+1}}$ $\quad y \equiv 0$ megoldás

Tehát $y(0)=\pi$ esetén $y(x) \equiv 0$. (3)

Ha $y \not\equiv 0$; szeparálható:

$$\int \frac{dy}{\operatorname{tg} y} = \int \frac{dx}{\sqrt{x^2+1}} \quad (2)$$

$$\int \frac{dy}{\operatorname{tg} x} = \int \frac{\cos y}{\sin y}\, dy \overset{f'/f \text{ alak}}{=} \ln|\sin y| + C \quad (2)$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \operatorname{arsh} x + C \quad (2)$$

$$\left.\begin{array}{l} \ln|\sin y| = \operatorname{arsh} x + C \\ y(0) = -\frac{\pi}{2} \end{array}\right\} \Longrightarrow 0 = C \Longrightarrow \underbrace{|\sin y|}_{-\sin y} = e^{\operatorname{arsh} x} \quad (2)$$

$$y(0) = -\frac{\pi}{2} \Longrightarrow y(x) = \arcsin\left(-e^{\operatorname{arsh} x}\right) \quad (2)$$

2, $\boxed{13}$ $y' + \frac{2}{x} y = \frac{\cos x}{x}$ $\quad$ elsőrendű lineáris

(H): $y' = -\frac{2}{x} y \Longrightarrow y \equiv 0$ mo.

Ha $y \not\equiv 0$: $\displaystyle\int \frac{dy}{y} = -2\int \frac{dx}{x} \Longrightarrow \ln|y| = -2\ln|x| + C \quad ; C \in \mathbb{R}$

$$y(x) = \pm e^C x^{-2}$$

Tehát $y_{H,\text{ált}}(x) = K \cdot \frac{1}{x^2}$; $K \in \mathbb{R}$ (5)

(I): Együttható variálás: $y_{I,p}(x) = \dfrac{k(x)}{x^2}$ (2) ; $y'_{I,p}(x) = \dfrac{k'(x)}{x^2} - 2\dfrac{k(x)}{x^3}$

Beírva: $\dfrac{k'(x)}{x^2} - 2\cancel{\dfrac{k(x)}{x^3}} + \cancel{\dfrac{2}{x} \cdot \dfrac{k(x)}{x^2}} = \dfrac{\cos x}{x} \Longrightarrow k'(x) = x \cos x$ (2)

$$k(x) = \int \underset{u}{x}\, \underset{v'}{\cos x} = x \sin x - \int 1 \cdot \sin x\, dx = x\sin x + \cos x \quad (2)$$

$u' = 1 \quad v = \sin x$

$y_{I,p}(x) = (x\sin x + \cos x) \cdot x^{-2}$; $\quad y_{I,\text{ált}}(x) = y_{H,\text{ált}}(x) + y_{I,p}(x) = \dfrac{K}{x^2} + \dfrac{x\sin x + \cos x}{x^2}$ (2)

$K \in \mathbb{R}$.

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3, $y' = \dfrac{1}{4x^2+y^2}$; $(x_0, y_0) = (1, 2)$

a, Érintő meredeksége: $\dfrac{1}{4\cdot 1^2+2^2} = \dfrac{1}{8}$ ②

$$\frac{1}{4x^2+y^2} = \frac{1}{8} \Rightarrow 4x^2+y^2 = 8 \Rightarrow \frac{x^2}{2} + \frac{y^2}{8} = 1$$ ellipszis mentén $\frac{1}{8}$ a meredekség. ④

b, Legyen $y(x)$ a (x_0, y_0)-on átmenő megoldás. $y'(x_0) = \frac{1}{8}$

$$y'' = \frac{-(8x+2y\cdot y')}{(4x^2+y^2)^2} \;④ ; \quad y''(x_0) = \frac{-8x_0 - 2y_0 y'(x_0)}{(4x_0^2+y_0^2)^2} = \underline{\underline{\frac{-8-1/2}{64}}} = \underline{\underline{\frac{-15}{128}}} \;②$$

$y''(x_0) \neq 0 \Rightarrow$ <u>Nincs inflexió!</u> ②

4, 15 $y' = \dfrac{5y-2x}{2x+y} = \dfrac{5(y/x)-2}{2+y/x}$ $\qquad u(x) = \dfrac{y(x)}{x}$ helyettesítés ②

$y(x) = u(x)\cdot x \Rightarrow y'(x) = u'(x)\cdot x + u(x)$

$$u'\cdot x + u = \frac{5u-2}{2+u} \Rightarrow u' = \frac{1}{x}\cdot\left(\frac{5u-2}{2+u} - u\right) = \frac{1}{x}\cdot\frac{-u^2+3u-2}{u+2}$$

$$\overbrace{\int \frac{u+2}{u^2-3u+2}\,du}^{I_1} = \underbrace{\int -\frac{1}{x}\,dx}_{I_2} \;④$$

$\left.\begin{array}{l} u = \frac{y}{x} \equiv 2 \text{ megold.} \\ u = \frac{y}{x} \equiv 1 \;\; -\text{"}- \end{array}\right\}$ ①

$u^2 - 3u + 2 = (u-2)(u-1)$

$$\frac{u+2}{u^2-3u+2} = \frac{A}{u-2} + \frac{B}{u-1};$$

$$u+2 = A(u-1) + B(u-2) = u\underbrace{(A+B)}_{1} + \underbrace{(-A-2B)}_{2}$$

$\left.\begin{array}{l} A+B=1 \\ -A-2B=2 \end{array}\right\}$ $\underline{B=-3;\ A=4}$

$$I_1 = \int\frac{4}{u-2}\,du + \int\frac{-3}{u-1}\,du = 4\ln|u-2| - 3\ln|u-1| + C \;⑥$$

$I_2 = -\ln|x| + C$ Figyelembe véve, hogy $u = \frac{y}{x}$, az impl. megold.:

$$\underline{\underline{4\ln\left|\frac{y}{x}-2\right| - 3\ln\left|\frac{y}{x}-1\right| = -\ln|x| + C}} \;② ; \quad \underline{\underline{y=2x}},\ \underline{\underline{y=x}}$$

5, $\boxed{11}$ $y = 3xe^{2x} - 4\sin x$

$\lambda_{1,2} = 2$; $\lambda_{3,4} = 0 \pm i$ (4)

Kar. pol.: $(\lambda-2)^2(\lambda+i)(\lambda-i) = (\lambda^2-4\lambda+4)(\lambda^2+1) =$

$= \lambda^4 - 4\lambda^3 + 5\lambda^2 - 4\lambda + 4$ (3)

diff. egy.: $y^{(4)} - 4y^{(3)} + 5y'' - 4y' + 4y = 0$ (2)

$y_{H,\text{ált}}(x) = Ae^{2x} + Bxe^{2x} + C\sin x + D\cos x$ (2)

6, $\boxed{14}$ $y''' - 2y'' + y' = \sin x + 2e^x$

(H) $\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda^2 - 2\lambda + 1) = \lambda(\lambda-1)^2 = 0$

$\Rightarrow \lambda_1 = 0,\ \lambda_{2,3} = +1$ (Belső rezonancia)

$y_{H,\text{ált}}(x) = C_1 + C_2 e^x + C_3 x e^x$; $C_1, C_2, C_3 \in \mathbb{R}$ (6p)

(I.) $y_{I,p}(x) = A\sin x + B\cos x + Cx^2e^x$ (3) külső rez.

$y'_{I,p}(x) = A\cos x - B\sin x + \underbrace{2Cxe^x + Cx^2e^x}_{Ce^x(2x+x^2)}$ $/\cdot(1)$

$y''_{I,p}(x) = -A\sin x - B\cos x + \underbrace{Ce^x(2x+x^2) + C\cdot(2+2x)e^x}_{Ce^x(x^2+4x+2)}$ $/\cdot(-2)$

$y'''_{I,p}(x) = -A\cos x + B\sin x + \underbrace{Ce^x(x^2+4x+2) + Ce^x(2x+4)}_{Ce^x(x^2+6x+2)}$ $/\cdot(1)$

(+)

$\sin x + 2e^x = \sin x(\cancel{-B} + 2A \cancel{+B}) + \cos x(\cancel{A} + 2B \cancel{-A}) +$

$+ Ce^x\underbrace{(2x + x^2 - 2(x^2+4x+2) + x^2+6x+2)}_{-2}$

$2A = 1$; $2B = 0$; $-2C = 2$

$A = \frac{1}{2}$; $B = 0$; $C = -1$ (3)

$$y_{I,p}(x) = \frac{1}{2}\sin x - x^2 e^x$$

$$y_{I,\text{ált}}(x) = y_{H,\text{ált}}(x) + y_{I,\text{part}}(x) = C_1 + C_2 e^x + C_3 x e^x + \frac{1}{2}\sin x - x^2 e^x \quad (2)$$

— o —

7, [10] Gyökkritérium (limeszes alak) (Többi alakot is elfogadjuk)

a, Ha $\forall n \in \mathbb{N}$ esetén $a_n > 0$, és $\exists \lim\limits_{n\to\infty} \sqrt[n]{a_n} < 1 \implies \sum\limits_{n=1}^{\infty} a_n < \infty$

b, Ha $\forall n \in \mathbb{N}$ esetén $a_n > 0$, és $\exists \lim\limits_{n\to\infty} \sqrt[n]{a_n} > 1 \implies \sum\limits_{n=1}^{\infty} a_n = \infty$ (4)

— o —

$$\sum_{n=1}^{\infty} \underbrace{\left(\frac{n-2}{n+3}\right)^{n^2}}_{0 < a_n} ; \quad \sqrt[n]{a_n} = \left(\frac{n-2}{n+3}\right)^n = \frac{\left(1-\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} \longrightarrow \frac{e^{-2}}{e^3} = e^{-5} < 1 \quad (4)$$

Tehát $\sum\limits_{n=1}^{\infty} a_n$ konvergens. (2)

— o —

8, [10] $f(n) = \frac{11}{2} f(n-1) + 3 f(n-2)$; $f(n) = q^n$

$$q^n = \frac{11}{2} q^{n-1} + 3 q^{n-2} \implies 2q^2 - 11q - 6 = 0 \quad (2)$$

$$q_{1,2} = \frac{11 \pm \sqrt{121 + 48}}{4} = \frac{11 \pm 13}{4} = \begin{cases} 6 \\ -\frac{1}{2} \end{cases} \quad (2)$$

$$f_{\text{ált}}(n) = A q_1^n + B q_2^n = A \cdot 6^n + B\left(-\frac{1}{2}\right)^n ; \quad A, B \in \mathbb{R} \quad (2)$$

Konvergens a megoldás, ha $A = 0$, B tetszőleges. (2)

Ekkor $f(0) = B$, $f(1) = -\frac{B}{2}$. Tehát $f(1) = -\frac{f(0)}{2}$ (2) esetén konvergens a megoldás.

Pótfeladatok: $\boxed{-5-}$

9, $\boxed{10}$

$$y^{(6)} - 16 y'' = 0$$

$$\lambda^6 - 16\lambda^2 \overset{②}{=} \lambda^2(\lambda^4 - 16) = \lambda^2(\lambda^2+4)(\lambda^2-4) =$$

$$= \lambda^2(\lambda+2i)(\lambda-2i)(\lambda+2)(\lambda-2) = 0 \quad ④$$

$$\lambda_{1,2} = 0\ ;\ \lambda_{3,4} = \pm 2i\ ;\ \lambda_5 = +2\ ;\ \lambda_6 = -2 \quad ②$$

$$y_{H,\acute{a}lt}(x) = C_1 + x C_2 + C_3 \sin(2x) + C_4 \cos(2x) + C_5 e^{2x} + C_6 e^{-2x} \quad ②$$

— ? —

10, $\boxed{10}$

$$\sum_{n=1}^{\infty} \underbrace{\frac{n!\,(4n)!}{(5n)!}}_{0 < a_n}$$

$$\frac{a_{n+1}}{a_n} \overset{②}{=} \frac{(n+1)!\,(4n+4)!}{(5n+5)!} \cdot \frac{(5n)!}{n!\,(4n)!} \; ② = \frac{(n+1)\,(4n+1)(4n+2)(4n+3)(4n+4)}{(5n+1)(5n+2)(5n+3)(5n+4)(5n+5)} \to$$

$$\to \frac{4^4}{5^5} \; ④ < 1 \Rightarrow \text{a sor } \underline{\underline{\text{konvergens}}} \; ②$$

(Két 5-ödfokú polinom hányadosának határértéke megegyezik a főegyütthatók hányadosával.)