2009. április 18.
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Fourier transformation

Fourier transform = complex spectrum

$$\mathcal{F}(x[k]) = X(e^{j\vartheta}) \quad \text{if} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\vartheta})\cdot e^{j\vartheta k}\, d\vartheta = x[k]$$

$$\mathcal{F}(x(t)) = X(j\omega) \quad \text{if} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot e^{j\omega t}\, d\omega = x(t)$$

if $x[k]$ is absolute summable: $X(e^{j\vartheta}) = \sum_{k=-\infty}^{\infty} x[k]\cdot e^{-j\vartheta k}$

if $x(t)$ is absolute integrable: $X(j\omega) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j\omega t}\, dt$

Response calculation based on Fourier transformation

(for BIBO stable, LI systems)

DT: $Y(e^{j\vartheta}) = H(e^{j\vartheta})\cdot U(e^{j\vartheta})$

CT: $Y(j\omega) = H(j\omega)\cdot U(j\omega)$

The Ft of the transfer char. equals to the Ft of the impulse response.

Rules of Fourier transformation

1, Linearity DT, CT: $\mathcal{F}(a x_1 + b x_2) = a\cdot\mathcal{F}(x_1) + b\cdot\mathcal{F}(x_2)$

2, Shifting DT: $\mathcal{F}(x[k-r]) = e^{-j\vartheta r}\cdot\mathcal{F}(x[k])$

CT: $\mathcal{F}(x(t-\tau)) = e^{-j\omega\tau}\cdot\mathcal{F}(x(t))$

$\left(\begin{matrix} X(e^{-j\vartheta}) = X^*(e^{j\vartheta}) \\ X(-j\omega) = X^*(j\omega)\end{matrix}\right)$

3, Derivative of CT signals $\mathcal{F}(x'(t)) = j\omega\cdot\mathcal{F}(x(t))$

4, Convolution theorem DT: $\mathcal{F}(u[k]*v[k]) = U(e^{j\vartheta})\cdot V(e^{j\vartheta})$

CT: $\mathcal{F}(u(t)*v(t)) = U(j\omega)\cdot V(j\omega)$

5, Modulation rule DT: $\mathcal{F}(x[k]\cdot e^{j\vartheta_0 k}) = \mathcal{F}(x[k])\Big|_{\vartheta\to\vartheta-\vartheta_0} = X(e^{j(\vartheta-\vartheta_0)})$

CT: $\mathcal{F}(x(t)\cdot e^{j\omega t}) = \mathcal{F}(x(t))\Big|_{\omega\to\omega-\omega_0} = X(j(\omega-\omega_0))$

Fourier transform of some special signals:

DT

$\mathcal{F}(\delta[k]) = 1 \quad (\mathcal{F}(\delta[k-r]) = e^{-jr\vartheta})$

$\mathcal{F}(\varepsilon[k]\cdot q^k) = \dfrac{1}{1-q\cdot e^{-j\vartheta}}$, $|q|<1$

CT

$\mathcal{F}(\delta(t)) = 1 \quad (\mathcal{F}(\delta(t-\tau)) = e^{-j\tau\omega})$

$\mathcal{F}(\varepsilon(t)\cdot e^{-\alpha t}) = \dfrac{1}{j\omega+\alpha}$, $\mathrm{Re}\,\alpha > 0$

Transfer char. – impulse response relation

DT: $h[k]$ is abs. summable $\Rightarrow$ $\mathcal{F}(h[k]) = H(e^{j\vartheta})$

CT: $h(t)$ is abs. integrable $\Rightarrow$ $\mathcal{F}(h(t)) = H(j\omega)$

DT: $\mathcal{F}(y[k]) = H(e^{j\vartheta})\cdot\mathcal{F}(u[k]) = H(e^{j\vartheta})\cdot U(e^{j\vartheta}) = Y(e^{j\vartheta})$ $\quad y[k] = \mathcal{F}^{-1}(Y(e^{j\vartheta}))$

CT: $\mathcal{F}(y(t)) = H(j\omega)\cdot\mathcal{F}(u(t)) = H(j\omega)\cdot U(j\omega) = Y(j\omega)$ $\quad y(t) = \mathcal{F}^{-1}(Y(j\omega))$

$$\cos x = \frac{e^{jx}+e^{-jx}}{2} \qquad \sin x = \frac{e^{jx}-e^{-jx}}{2j}$$

Distortionless signal transfer

ideal condition: ampl. char. const. & phase char. linearly decreasing

DT: $H(\vartheta) = K \qquad \varphi(\vartheta) = -\vartheta L \qquad (L\in\mathbb{N})$

CT: $H(\omega) = K \qquad \varphi(\omega) = -\omega T \qquad (T\in\mathbb{R}_+)$

Band limited CT signals

$x(t)$ is band limited, if $X(j\omega) = 0$ when $|\omega| > \Omega$

If a CT signal is band limited, it may be reconstructed from its samples if $T < \frac{\pi}{\underbrace{\Omega}_{\text{band limit}}}$

Fourier transform of periodic signals

DT: $\mathcal{F}(x[k]) = 2\pi \sum_{p = \begin{cases} -K \\ -K+1 \end{cases}}^{K} X_p^c \cdot \delta[\vartheta - p\vartheta_0]$

CT: $\mathcal{F}(x(t)) = 2\pi \sum_{p=-\infty}^{\infty} X_p^c \cdot \delta(\omega - p\omega_0)$