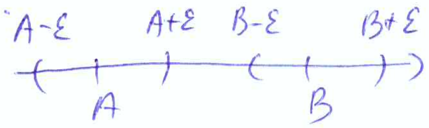
1, a) [3] D.: $\lim_{n\to\infty} a_n = A$, ha $\forall \varepsilon > 0$ esetén $\exists N(\varepsilon) \in \mathbb{N}$ küszöbindex, melyre $n > N(\varepsilon)$ esetén $|a_n - A| < \varepsilon$.

b) Indirekt bizonyítás. Tegyük fel, hogy $\lim_{n\to\infty} a_n = A$, és $\lim_{n\to\infty} a_n = B$,
[5] ahol $A < B$. Legyen $\varepsilon := \frac{B-A}{3} > 0$. Ekkor $\exists N_1(\varepsilon)$:

$$A - \varepsilon < a_n < A + \varepsilon \quad \forall n > N_1(\varepsilon) \text{ esetén}$$

és $\exists N_2(\varepsilon)$, melyre

$$B - \varepsilon < a_n < B + \varepsilon \quad \forall n > N_2(\varepsilon) \text{ esetén.}$$




Így ha $n > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$, akkor

$$a_n \in (A-\varepsilon, A+\varepsilon) \cap (B-\varepsilon, B+\varepsilon) = \emptyset \quad ↯$$

Ez ellentmondás, tehát az indirekt felt. hamis, az állítás igaz.

c) [5] $a_n = \frac{2n+3}{n+1} \xrightarrow{n\to\infty} 2$

$$|a_n - 2| = \left|\frac{2n+3-2(n+1)}{n+1}\right| = \frac{1}{n+1} < \varepsilon, \text{ ha } n > \frac{1}{\varepsilon} - 1, \quad (3)$$

tehát $N(\varepsilon) := \left[\frac{1}{\varepsilon}\right]$ jó küszöbindex. (2)

2, a) [6] T.: Ha $f$ az $I$ intervallumon invertálható, deriválható,
(2) és deriváltja nem nulla, akkor $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$.

(4) B.: $f^{-1}(f(x)) = x \quad / \frac{d}{dx}$

$$(f^{-1})'(f(x)) \cdot f'(x) = 1 \qquad y = f(x),\ x = f^{-1}(y)$$

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} \quad ✓$$

b) [5] def. alapján: $f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{\cancel{x} + \cancel{h} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}$ (5)

Inv. fv.: $g(x) = x^2$ az $f(x) = \sqrt{x}$ inverze, így
$f'(x) = \frac{1}{g'(f(x))} = \frac{1}{2\sqrt{x}}$ (5)

Csak a végeredmény: $f'(x) = \frac{1}{2\sqrt{x}}$ (1)

$-2-$

3, a, $\boxed{6}$ $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx = \int \frac{\cancel{t}}{t^2+1}\cdot\frac{1}{\cancel{t}}\,dt\Big|_{t=e^x}$ ③ $= (\operatorname{arctg} t + C)\Big|_{t=e^x}$ ② $=$

$t=e^x$

$dt = e^x dx = t\,dx$

$= \underline{\underline{\operatorname{arctg}(e^x) + C}}$ ①

b, $\boxed{6}$ $\displaystyle\int_{x=0}^{1} \sin^3(x)\,dx = \int_0^1 \sin(x)\underbrace{(1-\cos^2 x)}_{\sin^2 x}\,dx$ ③ $= \displaystyle\int_0^1 \sin x\,dx - \int \cos^2 x \sin x\,dx =$

$f^2\cdot f'$ alak

$= \left[-\cos x\right]_0^1 + \left[\dfrac{\cos^3 x}{3}\right]_0^1$ ② $= \underline{\underline{-\cos 1 + 1 + \dfrac{\cos^3 1}{3} - \dfrac{1}{3}}}$ ①

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4, $\boxed{12}$ $y'(x) = \dfrac{xy}{x^2+y^2} = \dfrac{y/x}{1+(y/x)^2}$ ; $u = \dfrac{y}{x}$ ; $y(x) = x\cdot u(x)$ ②

$y'(x) = u(x) + x u'(x)$

$u + xu' = \dfrac{u}{1+u^2}$ ②

$u' = \dfrac{1}{x}\left(\dfrac{u}{1+u^2} - u\right) = \dfrac{1}{x}\left(\dfrac{\cancel{u}-\cancel{u}-u^3}{1+u^2}\right)$ Szeparálható

$\displaystyle\int \underbrace{\frac{1+u^2}{u^3}}_{u^{-3}+u^{-1}}\,du = -\int \frac{dx}{x}$ ② $\Longrightarrow -\dfrac{u^{-2}}{2} + \ln|u| = -\ln|x| + C$ ② ②

$\underline{\underline{-\dfrac{x^2}{2y^2} + \ln\left|\dfrac{y}{x}\right| = -\ln|x| + C}}$ ②

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5, a, $\boxed{6}$ $S_n = \displaystyle\sum_{k=1}^{n} a_1\cdot q^{k-1}$ ; $qS_n = \displaystyle\sum_{k=1}^{n} a_1\cdot q^k$

$S_n - qS_n = (1-q)S_n = (a_1 + \cancel{a_1 q} + \cancel{a_1 q^2} + \dots + \cancel{a_1 q^{n-1}} - \cancel{a_1 q} - \cancel{a_1 q^2} - \dots - a_1 q^n) =$

$= a_1 - a_1 q^n \Longrightarrow \underline{\underline{S_n = a_1\dfrac{1-q^n}{1-q}}}$

tagok száma

hányados

első tag

b, $\boxed{4}$ $\displaystyle\sum_{n=3}^{\infty} \frac{2^{2n+1}}{5^n} = 2\sum_{n=3}^{\infty}\left(\frac{4}{5}\right)^n = \underline{\underline{2\cdot\left(\frac{4}{5}\right)^3\cdot\frac{1}{1-\frac{4}{5}}}}$ ④ $= 2\cdot\dfrac{4^3}{5^3}\cdot 5 = \underline{\underline{\dfrac{128}{25}}}$

6, $\boxed{10}$

$$f(x)=\frac{1}{\sqrt{4+x^3}}=(4+x^3)^{-1/2}=4^{-1/2}\left(1+\frac{x^3}{4}\right)^{-1/2} \ (2) = \frac{1}{2}\sum_{k=0}^{\infty}\binom{-1/2}{k}\frac{x^{3k}}{4^k} \ (3)$$

binomiális sorfejtés.

$x$-nek legfeljebb 8. hatványát tartalmazó tagok összege $T_8(x)$:

$$T_8(x)=\frac{1}{2}\binom{-1/2}{0}\frac{x^0}{4^0}+\frac{1}{2}\binom{-1/2}{1}\frac{x^3}{4^1}+\frac{1}{2}\binom{-1/2}{2}\frac{x^6}{4^2} \ (2) =$$

$$(3) \ =\frac{1}{2}+\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\frac{x^3}{4}+\frac{1}{2}\,\frac{(-1/2)(-3/2)}{1\cdot 2}\cdot\frac{x^6}{4^2}=\underline{\underline{\frac{1}{2}-\frac{x^3}{16}+\frac{3x^6}{128}}}$$

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7, a, $\boxed{4}$ D: $f(x,y)$ totálisan differenciálható az $\underline{r}_0=(x_0,y_0)$ pontban, ha

$\exists\ \underline{A}=\begin{bmatrix}A_x\\A_y\end{bmatrix}\in\mathbb{R}^2$ úgy, hogy

$$f(x_0+\Delta x,y_0+\Delta y)=f(x_0,y_0)+\underbrace{A_x\cdot\Delta x+A_y\cdot\Delta y}_{\underline{A}\cdot\Delta\underline{r}}+\underline{\varepsilon}(x,y)\cdot\Delta\underline{r},$$

ahol $\lim\limits_{\Delta\underline{r}\to 0}\underline{\varepsilon}(x,y)=0$.

b, T.: Ha $f$ totálisan deriválható az $(x_0,y_0)$ pontban, akkor

$\boxed{4}$ $\exists\ f'_x(x_0,y_0)$ (1), $\exists\ f'_y(x_0,y_0)$ (1), és $f$ folytonos (2) az $(x_0,y_0)$ pontban.

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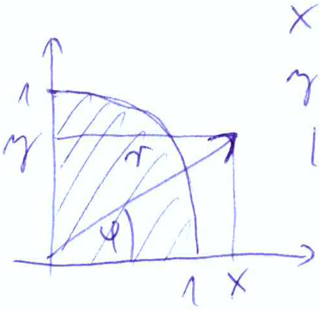
8, $f(x,y)=x^2y$

$\boxed{12}$




$x=r\cos\varphi$, $\quad 0\le r\le 1$

$y=r\sin\varphi$, $\quad 0\le\varphi\le\frac{\pi}{2}$ $\Big\}\ T$ (2)

$|J|=r$

$$\int_T f(x,y)\,dT=\int_{r=0}^{1}\int_{\varphi=0}^{\pi/2}(r\cos\varphi)^2\cdot r\sin\varphi\cdot\underbrace{r\,d\varphi\,dr} \ (3) =\int_{r=0}^{1}\int_{\varphi=0}^{\pi/2}r^4\cos^2\varphi\sin\varphi\,d\varphi\,dr=$$

$$=\left(\int_{r=0}^{1}r^4\,dr\right)\cdot\left(\int_{\varphi=0}^{\pi/2}\cos^2\varphi\sin\varphi\,d\varphi\right) (2) =\left[\frac{r^5}{5}\right]_0^1 (2) \cdot\left[-\frac{\cos^3\varphi}{3}\right]_0^{\pi/2} (2) =$$

$$=\frac{1}{5}\cdot\frac{0-(-1)}{3}=\underline{\underline{\frac{1}{15}}} \ (1)$$

9, a, $\boxed{6}$

③ D: $(f * g)(x) = \int_{t=-\infty}^{\infty} f(t)\, g(x-t)\, dt$ (konvolúció)

③ T.) $\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$

b, A definíció alapján:

⑥ $\mathcal{F}[g](\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(2(x+3))\, dx = \int_{-\infty}^{\infty} e^{-i\omega(\frac{t}{2}-3)} f(t) \cdot \frac{1}{2}\, dt =$

$t := 2(x+3)$

$dt = 2dx\,;\ x = \frac{t}{2} - 3$

$= \frac{1}{2} \cdot e^{3i\omega} F\left(\frac{\omega}{2}\right)$

Vagy a szabályok alkalmazásával:

⑥ $\mathcal{F}[g](\omega) = \mathcal{F}[f(2(x+3))](\omega) = e^{3i\omega} \mathcal{F}[f(2x)](\omega) = e^{3i\omega} \frac{1}{2} F\left(\frac{\omega}{2}\right)$