1, a, [6] Tétel: Ha $a_n \to A$, és $b_n \to A$, és $\forall n \in \mathbb{N}$ esetén $a_n \le c_n \le b_n$, akkor $c_n \to A$. ②

B.: $\forall \varepsilon > 0$ esetén $\exists N_a(\varepsilon)$: $A-\varepsilon < a_n < A+\varepsilon$, ha $n > N_a(\varepsilon)$, s

$\exists N_b(\varepsilon)$: $A-\varepsilon < b_n < A+\varepsilon$, ha $n > N_b(\varepsilon)$.

Tehát ha $n > N_c(\varepsilon) := \max\{N_a(\varepsilon), N_b(\varepsilon)\}$, akkor

$A-\varepsilon < a_n \le c_n \le b_n < A+\varepsilon$, ami azt jelenti, hogy $c_n \to A$. ④

b, [6]

$$② \underbrace{\frac{e^3}{\sqrt[n]{7}\cdot\sqrt[n]{n}}}_{\downarrow\, e^3} = \sqrt[n]{\frac{e^{3n}}{2n+5n}} \le \underbrace{\sqrt[n]{\frac{e^{3n}+n^2}{2n+5}}}_{\downarrow\, n\to\infty\ e^3} \le \sqrt[n]{\frac{e^{3n}+e^{3n}}{1}} = \underbrace{\sqrt[n]{2}\cdot e^3}_{\downarrow\, e^3} ②$$

Tehát a rendőr-elv alapján a határérték: $\underline{\underline{e^3}}$. ②

2, a, [7] $(f\cdot g)' = f'g + fg'$ ②

$$(fg)'(x) = \lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} =$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} =$$

$$= \lim_{h\to 0}\underbrace{g(x+h)}_{\downarrow\, g(x)}\cdot\underbrace{\frac{f(x+h)-f(x)}{h}}_{\downarrow\, f'(x)} + \underbrace{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}_{g'(x)}\cdot f(x) = (f'g+fg')(x)$$ ⑤

b, [5]

$$\frac{d}{dx}\left(\frac{e^{3x^2+2x}}{\sqrt{x^2+4}}\right) = \frac{e^{3x^2+2x}\cdot(6x+2)\cdot\sqrt{x^2+4} - e^{3x^2+2x}\cdot\frac{1}{2}(x^2+4)^{-1/2}\cdot 2x}{x^2+4}$$

3, a, $\boxed{6}$ $\displaystyle\int x\cdot\sqrt{1-x^2}\,dx = \int \sin\varphi\cdot\cos^2\varphi\,d\varphi\Big|_{\varphi=\arcsin x} = -\frac{\cos^3\varphi}{3}\Big|_{\varphi=\arcsin x} + C =$

$x=\sin\varphi$
$dx=\cos\varphi\,d\varphi$

$$= -\frac{1}{3}\cdot\cos^3(\arcsin x) + C = \underline{\underline{-\frac{1}{3}\left(1-x^2\right)^{3/2}+C}}$$

vagy $\displaystyle\int x\sqrt{1-x^2}\,dx = \frac{-1}{2}\cdot\int(-2x)(1-x^2)^{1/2}\,dx = \underline{\underline{-\frac{1}{2}\cdot(1-x^2)^{3/2}\cdot\frac{2}{3}+C}}$

$f'\cdot f^{1/2}$ alak

b, $\boxed{6}$ $\displaystyle\int_0^1 2\cdot x^{-1/3}\,dx = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 2x^{-1/3}\,dx = \lim_{\varepsilon\to0+} 2\left[\frac{3}{2}x^{2/3}\right]_\varepsilon^1 = \underline{\underline{3}}$

— o —

6, Elsőrendű lineáris egyenletről van szó. A homogén egyenlet:

$\boxed{12}$ $y' + \dfrac{y}{x} = 0 \implies \displaystyle\int\frac{dy}{y} = -\int\frac{dx}{x} \implies \ln|y| = -\ln|x| + C$;

$\underline{y_{H,\text{ált}}(x) = A\cdot\dfrac{1}{x}}$ 4

Az inhomogén egyenletet az állandó variálásával oldjuk meg:

$y_{I,p}(x) = A(x)\cdot\dfrac{1}{x}$; $y'_{I,p}(x) = \dfrac{A'(x)}{x} - \dfrac{A(x)}{x^2}$, így

$\dfrac{A'}{x} - \cancel{\dfrac{A}{x^2}} + \cancel{\dfrac{A}{x^2}} = \dfrac{x+1}{x}\cdot e^x \implies A'(x) = (x+1)\cdot e^x$;

$A(x) = \displaystyle\int(x+1)\,e^x\,dx = (x+1)\,e^x - \int e^x\,dx = x\,e^x$;

$\underline{y_{I,p}(x) = \dfrac{A(x)}{x} = e^x}$ 4 ; $y_{I,\text{ált}}(x) = y_{I,p} + y_{H,\text{ált}} = \underline{e^x + \dfrac{A}{x}}$ 2

Kezdeti felt. illesztése: $e^1 + \dfrac{A}{1} = e+1 \implies A = 1$, $\underline{\underline{y(x) = e^x + \dfrac{1}{x}}}$ 2

5, a, $\boxed{4}$ A $\sum_{n=0}^{\infty} a_n$ sor feltételesen konvergens, ha konvergens, de nem abszolút konvergens (4), azaz $\exists \lim_{n\to\infty} \sum_{k=0}^{n} a_k = \lim_{n\to\infty} S_n = S \in \mathbb{R}$, és

$\sum_{n=0}^{\infty} |a_n| = \infty$.

b, $\boxed{8}$ $\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{n^2}{n^3+1}}_{c_n}$ A sor Leibniz-típusú, hiszen alternáló ✓ (1)

$c_n = \frac{n^2}{n^3+1} = \frac{1}{\underset{\infty}{n}+1} \xrightarrow{n\to\infty} 0$, és (1)

$c_{n+1} \overset{?}{<} c_n$; $\frac{(n+1)^2}{(n+1)^3+1} \overset{?}{<} \frac{n^2}{n^3+1}$

$(n^2+2n+1)(n^3+1) \overset{?}{<} ((n^3+3n^2+3n+1)+1)\cdot n^2$

$\cancel{n^5}+2n^4+n^3+n^2+2n+1 \overset{?}{<} \cancel{n^5}+3n^4+3n^3+2n^2$

$0 < n^4+2n^3+n^2-2n-1$ ✓ igaz, ha $n \geq 1$. (2)

Így a sor konvergens. (1)

A sor nem abszolút konv., mert $\sum_{n=1}^{\infty} \frac{n^2}{n^3+1} \geq \sum_{n=1}^{\infty} \frac{n^2}{n^3+n^3} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ (2)

Tehát a sor feltételesen konv. (1)

6, $\boxed{8}$ $f(x) = \frac{1}{2x+3} = \frac{1}{2(x-1)+5} = \frac{1}{5} \frac{1}{1-\left(\frac{-2(x-1)}{5}\right)} = \frac{1}{5} \sum_{k=0}^{\infty} \left(\frac{-2}{5}\right)^k (x-1)^k =$

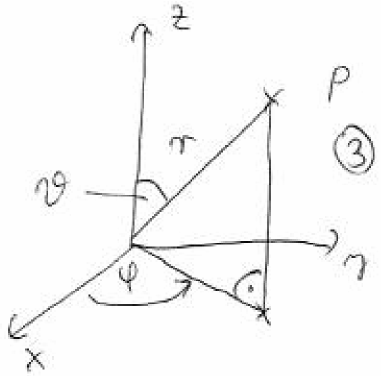
$= \sum_{k=0}^{\infty} \frac{(-2)^k}{5^{k+1}} (x-1)^k$, (5) ha $\left|\frac{-2(x-1)}{5}\right| < 1$, azaz $|x-1| < \frac{5}{2}$, tehát

a konvergenciatartomány: $K.T. = \left(-\frac{3}{2}, \frac{7}{2}\right)$ (3)

7, $\boxed{8}$ a, $f'_x(x,y) = \frac{\partial}{\partial x}\left(g\left(\frac{3y}{1+x^2}\right)\right) = g'\left(\frac{3y}{1+x^2}\right)\cdot\frac{-6xy}{(1+x^2)^2}$ (4)

b, $f''_{xy}(x,y) = \frac{\partial}{\partial y}\left(f'_x(x,y)\right) = g''\left(\frac{3y}{1+x^2}\right)\cdot\frac{-18xy}{(1+x^2)^3} + g'\left(\frac{3y}{1+x^2}\right)\cdot\frac{-6x}{(1+x^2)^2}$ (4)

8, a, $\boxed{6}$



(3)

$x = r\sin\vartheta\cos\varphi$ (1)

$y = r\sin\vartheta\sin\varphi$ (1)

$z = r\cos\vartheta$ (1)

b, $\boxed{6}$ $$V(R) = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi}\int_{r=0}^{R} 1\cdot(r^2\sin\vartheta)\,dr\,d\vartheta\,d\varphi = 2\pi\cdot\int_{\vartheta=0}^{\pi}\sin\vartheta\,d\vartheta\cdot\int_{r=0}^{R} r^2\,dr =$$ (2)

$$= 2\pi\cdot\underbrace{\Big[-\cos\vartheta\Big]_0^{\pi}}_{2}\cdot\underbrace{\Big[\frac{r^3}{3}\Big]_0^{R}}_{\frac{R^3}{3}} = \frac{4\pi}{3}R^3$$ (4)

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9 a, $\boxed{6}$ T.: $\mathcal{F}[f(x+a)](\omega) = e^{i\omega a}\mathcal{F}[f](\omega)$ (2)

B.: $$\mathcal{F}[f(x+a)](\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(\underbrace{x+a}_{y})\,dx = \int_{-\infty}^{\infty} e^{-i\omega(y-a)} f(y)\,dy =$$

$$= e^{i\omega a}\int_{-\infty}^{\infty} e^{-i\omega y} f(y)\,dy = e^{i\omega a}\mathcal{F}[f](\omega)$$ (4)

b, $\boxed{6}$ $$\mathcal{F}[g](\omega) = \mathcal{F}[f(2x+3)](\omega) = \frac{1}{2}\mathcal{F}[f(x+3)]\left(\frac{\omega}{2}\right) = \frac{1}{2}e^{i\frac{\omega}{2}\cdot 3}F\left(\frac{\omega}{2}\right)$$

Vagy a def. alapján, helyettesítéssel:

$$\mathcal{F}[g](\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(\underbrace{2x+3}_{y=2x+3})\,dx = \int_{-\infty}^{\infty} e^{-i\omega\left(\frac{y-3}{2}\right)} f(y)\cdot\frac{1}{2}\,dy =$$

$x = \frac{y-3}{2}$; $dx = \frac{1}{2}dy$

$$= \frac{1}{2}\cdot e^{\frac{3}{2}i\omega}\cdot\int_{-\infty}^{\infty} e^{-i\frac{\omega}{2}y} f(y)\,dy = \frac{1}{2}\cdot e^{\frac{3}{2}i\omega}F\left(\frac{\omega}{2}\right)$$