Laurent Series

it is often necessary to expand a function $f(z)$ around points at which it is no longer analytic, but is singular

→ new type of series : Laurent series, consisting of positive and negative integer powers of $(z - z_0)$,
converges in some annulus (körgyűrű) centered at z_0 in which $f(z)$ is analytic

Laurent's theorem

If $f(z)$ is analytic on two concentric circles C_1 and C_2 with center at z_0 and in the annulus D between them, then

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}}, \quad \text{where}$$

the coefficients:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*) dz^*,$$

each integral being taken counterclockwise around any simple closed path C in D

This series converges and represents $f(z)$ in the open annulus obtained from the given one by continuously increasing C_1 and decreasing C_2 until each of the two circles reaches a point where $f(z)$ is singular.

Special case: when the only singular point of $f(z)$ is z_0 → series converges in a disk except at the center

Notation:

$$(*) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \qquad b_n = a_{-n}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*$$

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f: (A) $n = -1$: $\oint (*) \to \oint_C f(z) dz = \sum a_n \underbrace{\oint (z - z_0)^n}_{\substack{= 2\pi i \text{ if } n = -1 \\ = 0 \text{ if } n \neq -1}} = a_{-1} 2\pi i \longrightarrow a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz$

↑ uniformly converge, termwise integration

(B) $n = 0$: $\oint \frac{(*)}{z - z_0} \to \oint_C \frac{f(z)}{z - z_0} dz = \sum a_n \underbrace{\oint (z - z_0)^{n-1} dz}_{\substack{2\pi i, \text{ if } n = 0 \\ 0 \text{ if } n \neq 0}} \longrightarrow a_0 = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$

(C) $n = -2$: $\oint (*) \cdot (z - z_0) dz \to \oint_C f(z)(z - z_0) dz = \sum a_n \underbrace{\oint (z - z_0)^{n+1} dz}_{\substack{2\pi i \text{ if } n = -2 \\ 0 \text{ if } n \neq -2}} — a_{-2} = \frac{1}{2\pi i} \oint_C f(z)(z - z_0)$

⋮

Ex:

1. Develop $f(z) = \frac{1}{1-z}$ in a.) nonnegative powers of z
b.) negative powers of z

a.) $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ if $|z| < 1$

b.) $\frac{1}{1-z} = \frac{-1}{z(1-\frac{1}{z})} = -\sum_{n=0}^{\infty} \frac{1}{z} \cdot \left(\frac{1}{z}\right)^n = -\frac{1}{z} - \frac{1}{z^2} - \dots$, if $|z| > 1$

substitution

2. Find the Laurent series of $z^{-5} \cdot \sin z$ with center $z = 0$

$$z^{-5} \sin z = z^{-5}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \frac{1}{z^4} - \frac{1}{z \cdot 3!} + \frac{1}{5!} - \frac{z^2}{7!} + \dots$$

, if $|z| > 0$

3. Find the Laurent series of $z^2 e^{1/2}$, $z_0 = 0$

$$z^2 \cdot e^{1/z} = z^2 \cdot \left(1 + \frac{1}{1!\, z} + \frac{1}{2!\, z^2} + \dots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!\, z} + \frac{1}{4!\, z^2} + \dots$$

$|z| > 0$

4. Find all Laurent series of $\frac{1}{z^3 - z^4}$; $z_0 = 0$

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3(1-z)} = \sum_0^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \dots \qquad 0 < |z| < 1$$

$$\frac{1}{z^3 - z^4} = -\frac{1}{z^4(1-\frac{1}{z})} = -\sum_0^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \dots \qquad |z| > 1$$

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5. Find Taylor and Laurent series of $f(z) = \frac{-2z+3}{z^2 - 3z + 2}$ with center 0

$f(z) = \underbrace{-\frac{1}{z-1}}_{\text{see Ex 1.}} - \frac{1}{z-2}$

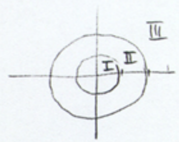
(A) $-\frac{1}{z-2} = \frac{1}{2(1-\frac{1}{2}z)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}z\right)^n \qquad |z| < 2$

(B) $-\frac{1}{z-2} = -\frac{1}{z(1-\frac{2}{z})} = -\sum_0^{\infty}\left(\frac{2}{z}\right)^n \cdot \frac{1}{z} \qquad |z| > 2$

I $|z| < 1$ $\quad f(z) = \sum z^n + \sum \frac{z^n}{2^{n+1}} = \sum\left(1 + \frac{1}{2^{n+1}}\right) z^n$

II $1 < |z| < 2$ $\quad f(z) = -\sum \frac{1}{z^{n+1}} + \sum \frac{z^n}{2^{n+1}}$

III $2 < |z|$ $\quad f(z) = -\sum \frac{1}{z^{n+1}} - \sum \frac{2^n}{z^{n+1}} = -\sum (1 + 2^n) \frac{1}{z^{n+1}}$



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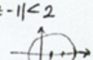
6. Find the Laurent series of $f(z) = \frac{1}{1-z^2}$ that converges in the annulus $\frac{1}{4} < |z-1| < \frac{1}{2}$ and determine the precise region of convergence

$z_0 = 1 \rightarrow$ develop $f(z)$ in powers of $z - 1$. $\rightarrow f(z) = \frac{-1}{(z-1)(z+1)}$

$\frac{1}{z+1} = \frac{1}{2 + (z-1)} = \frac{1}{2} \frac{1}{1 - (-\frac{z-1}{2})} = \frac{1}{2}\sum\left(\frac{-(z-1)}{2}\right)^n = \sum_0^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^n$ converges $\left|\frac{z-1}{2}\right| < 1 \rightarrow |z-1| < 2$

Multiply by $\frac{-1}{z-1} \rightarrow f(z) = -\sum_0^{\infty} \frac{(-1)^n}{2^{n+1}} (z-1)^{n-1}$, region of convergence $|z-1| < 2$



If $f(z)$ in Laurent's theorem is analytic inside C_2, the coefficients b_n are zero by Cauchy's int. theo.
$\rightarrow$ the Laurent series reduces to a Taylor series (see ex. 5 I)

Singularities and Zeros. Classification of singularities

def: $z=z_0$ is an <u>isolated singularity</u> of $f(z)$ if $z=z_0$ has a neighbourhood without further singularities of $f(z)$

ex: $\tan z$ has isolated singularities at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$\tan\frac{1}{z}$ has a nonisolated singularity at 0

$$(1) \quad f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic at } z=z_0} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part of (1)}}$$

If the principal part has only finitely many terms, it is of the form

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}.$$

def: the singularity of $f(z)$ at $z=z_0$ is called a <u>pole</u>, and m is called its <u>order</u>

(Poles of first order = simple poles)

def: if the principal part of (1) has infinitely many terms, we say $f(z)$ has at $z=z_0$ an <u>isolated essential singularity</u>.

ex: $f(z) = \frac{1}{z(z-2)^5} + \frac{1}{(z-2)^5}$ has a simple pole at $z=0$, a pole of fifth order at $z=2$

ex: functions having an isolated essential singularity:

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots$$

<u>Theorem 1</u>: If $f(z)$ is analytic and has a pole at $z=z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ in any manner.

def: $f(z)$ has a <u>removable singularity</u> at $z=z_0$ if $f(z)$ is not analytic at $z=z_0$ but can be made analytic there by assigning a suitable value $f(z_0)$

ex: $f(z) = \frac{\sin z}{z}$ becomes analytic at $z=0$ if we define $f(0)=1$

If we want to investigate $f(z)$ for large $|z| \to z = \frac{1}{w} \to f(z) = f(\frac{1}{z}) = g(w)$

$\to$ investigate $g(w)$ at $w=0$

$g(w)$ is singular at $w=0 \to f(z)$ is singular at ∞

analytic — " — $f(z)$ is analytic at ∞

Residues

→ can be used to evaluate integrals $\oint_C f(z)\,dz$

- if $f(z)$ is analytic on C and inside C → $\oint_C f(z)\,dz = 0$ (Cauchy's int. thec
- if $f(z)$ is analytic on C and inside C except at $z = z_0$ ⇒ Laurent serie:

$$f(z) = \sum_0^\infty a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots ,$$

that converges for all points near z_0 (except at $z = z_0$ itself) in some domain $0 < |z - z_0| < R$

$$b_1 = \frac{1}{2\pi i} \oint_C f(z)\,dz \quad \rightarrow \quad \boxed{\oint f(z)\,dz = 2\pi i\, b_1}$$

↑ can be found in another way

$$\boxed{b_1 = \operatorname*{Res}_{z=z_0} f(z)}$$

evaluation of an integral by means of a residue:

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \frac{z^7}{7!} + \dots , \quad \text{converges for } |z| > 0$$

↳ has a pole of third order at $z = 0$ → $b_1 = -\frac{1}{3!}$

$$\rightarrow \oint_{|z|=1} \frac{\sin z}{z^4}\,dz = 2\pi i \left(-\frac{1}{3!}\right) = \underline{\underline{-\frac{\pi i}{3}}}$$

FORMULA 1

Do we need the whole series to find b_1? → No longer

Let $f(z)$ have a simple pole at $z = z_0$

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \qquad 0 < |z-z_0| < R$$

$$(z-z_0) f(z) = b_1 + (z-z_0)\left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots\right]$$

$\xrightarrow{\text{let } z \to z_0}$ $$\boxed{\operatorname*{Res}_{z=z_0} f(z) = b_1 = \lim_{z\to z_0} (z-z_0) f(z)}$$

ex: $$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+1)} = \lim_{z\to i} (z-i)\frac{9z+i}{z(z+i)(z-i)} = \left.\frac{9z+i}{z(z+i)}\right|_{z=i} = \frac{10i}{-2} = \underline{\underline{-5i}}$$

FORMULA 2

if $f(z) = \frac{p(z)}{q(z)}$ ← analytic, $p(z_0) \neq 0$, $q(z)$ has a simple zero at $z = z_0$ → Taylor

$$\rightarrow q(z) = (z-z_0)\, q'(z_0) + \frac{(z-z_0)^2}{2!} q''(z_0) + \dots$$

$$\operatorname{Res} f(z) = \lim_{z\to z_0} (z-z_0)\frac{p(z)}{q(z)} = \lim_{z\to z_0} \frac{\cancel{(z-z_0)}\, p(z)}{\cancel{(z-z_0)}\left[q'(z_0) + \frac{z-z_0}{2!} q''(z_0) + \dots\right]} = \frac{p(z_0)}{q'(z_0)}$$

$$\boxed{\operatorname*{Res}_{z\to z_0} f(z) = \operatorname*{Res}_{z\to z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}}$$

1.) example:

$$f(z) = \frac{\cosh \pi z}{z^4 - 1} \quad \leftarrow \text{entire } p(z)$$

$z^4-1 \rightarrow$ has simple zeros at $1, -1, i, -i$
$q(z)$

$$q'(z) = 4z^3$$

$$\operatorname*{Res}_{1} f(z) = \underline{\underline{\frac{\cosh \pi}{4}}} \quad , \quad \operatorname*{Res}_{-1} f(z) = \frac{\cosh(-\pi)}{-1 \cdot 4} = \underline{\underline{-\frac{\cosh \pi}{4}}} \quad !$$

$$\operatorname*{Res}_{i} f(z) = \frac{\cosh \pi i}{-4i} = \frac{\cos \pi}{-4i} = \frac{-1}{-4i} = \underline{\underline{\frac{-i}{4}}} \quad , \quad \operatorname*{Res}_{-i} f(z) = \frac{\cosh(-\pi i)}{4(-i)^3} = \frac{\cos(-\pi)}{4i} = \frac{-1}{4i} =$$

2.) example: $f(z) = \frac{1}{z^2 + 4iz} = \frac{1}{z(z+4i)}$, singularities $z=0$, $z=-4i$

Expand L.S at a.) $z=0$, b.) $z=-4i$, find $\operatorname*{Res}_{0} f(z)$, $\operatorname*{Res}_{-4i} f(z)$

a.) $z = 0 \rightarrow f(z) = \frac{1}{z} \frac{1}{z+4i} = \frac{1}{z} \cdot \frac{1}{4i} \frac{1}{1+\frac{z}{4i}} = \frac{1}{4i} \frac{1}{z} \left(1 - \frac{z}{4i} + \left(\frac{z}{4i}\right)^2 - \left(\frac{z}{4i}\right)^3 - \dots\right)$

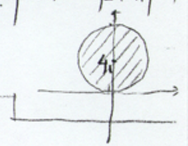
valid if $\left|\frac{z}{4i}\right| < 1 \rightarrow |z| < 4$

$$\operatorname*{Res}_{0} f(z) = \frac{1}{4i}$$

b.) $z = -4i \rightarrow f(z) = \frac{1}{z+4i} \cdot \frac{1}{z} = \frac{1}{z+4i} \frac{1}{z+4i-4i} = -\frac{1}{4i} \frac{1}{z+4i} \frac{1}{1 - \frac{z+4i}{4i}} =$

$= -\frac{1}{4i} \frac{1}{z+4i} \left(1 + \frac{z+4i}{4i} + \left(\frac{z+4i}{4i}\right)^2 + \dots\right)$, valid if $\left|\frac{z+4i}{4i}\right| < 1 \rightarrow |z+4i| < 4$

$$\operatorname*{Res}_{-4i} f(z) = -\frac{1}{4i}$$



Residue Theorem

We shall now see that the residue integration method can be extended to the case of several singular points of $f(z)$ inside C

Theorem:

Let $f(z)$ be a function that is analytic inside a simple closed path C and on C, except for finitely many singular points $z_1, z_2, \dots, z_k$ inside C

$$\rightarrow \boxed{\oint f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)}$$

integral being taken counterclockwise

Proof:

$$\left.\begin{aligned} &\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \dots + \oint_{C_k} f(z)\,dz = 0 \\ &\text{Cauchy's int. theo.} \\ &\oint_{C_j} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_j} f(z) \end{aligned}\right\} \text{q.e.d}$$

Examples : integration by residue theorem

3) Evaluate the following integral counterclockwise around any simple closed path such that

a.) 0 and 1 are inside C

b) 0 is inside, 1 outside

c.) 1 is inside, 0 outside

d.) 0 and 1 are outside

$$\oint_C \frac{4-3z}{z^2-z}\,dz = ?$$

integrand has simple poles at 0 and 1

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z(z-1)} = \left.\frac{4-3z}{z-1}\right|_{z=0} = \frac{4}{-1} = -4$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z(z-1)} = \left.\frac{4-3z}{z}\right|_{z=1} = \frac{1}{1} = 1$$

a.) $\oint f(z)dz = 2\pi i(-4+1) = 6\pi i$

b.) $\oint f(z)dz = 2\pi i(-4) = -8\pi i$

c.) $\oint f(z)dz = 2\pi i \cdot 1 = 2\pi i$

d.) $\oint f(z)dz = 0$ ($\leftarrow$ Cauchy's Theorem)

4.) $$\oint_{|z|=6} \frac{1}{z^2+4iz}\,dz \underset{\text{Res.Theo}}{=} 2\pi i\left(\operatorname*{Res}_{0} f + \operatorname*{Res}_{-4i} f\right) = 2\pi i\left(\frac{1}{4i} - \frac{1}{4i}\right) = \underline{\underline{0}}$$

singularities : $z=0$, $z=-4i$ (look at ex. 2.))

$\hookrightarrow$ $$\operatorname*{Res}_{0} \frac{1}{z^2+4iz} = \frac{1}{4i}$$

$$\operatorname*{Res}_{-4i} \frac{1}{z^2+4iz} = -\frac{1}{4i}$$