A3 – 4. Vizsga megoldásai (2015.01.21.)

① $y''(x) - 2y'(x) + y(x) = \dfrac{e^x}{x} \qquad x \neq 0$

H: $\quad y'' - 2y + y = 0 \quad \rightsquigarrow$ kar. polinom: $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$

$\hookrightarrow \lambda_1 = \lambda_2 = 1$ (kétszeres)

$\hookrightarrow \boxed{y_H = C_1 e^x + C_2 x e^x}$ $\boxed{3p}$

IH: állandók variálásának módszere:

$$y_p = C_1(x) e^x + C_2(x) x e^x$$

$$y_p' = C_1'(x) e^x + C_1(x) e^x + C_2'(x) x e^x + C_2(x)(e^x + x e^x) =$$

$$= \underbrace{C_1'(x) e^x + C_2'(x) x e^x}_{\text{tfh } = 0} + e^x (C_1(x) + C_2(x)) + C_2(x) x e^x$$

$$y_p'' = e^x (C_1(x) + C_2(x)) + e^x (C_1'(x) + C_2'(x)) + C_2'(x) x e^x +$$

$$+ C_2(x)(e^x + x e^x) = e^x (C_1 + 2C_2 + C_1' + C_2') + x e^x (C_2 + C_2')$$

$\boxed{3p}$

behelyettesítve

$$e^x (C_1 + 2C_2 + C_1' + C_2') + x e^x (C_2 + C_2') - 2e^x (C_1 + C_2) - 2C_2 x e^x +$$

$$+ C_1 e^x + C_2 x e^x = \frac{e^x}{x}$$

$\hookrightarrow \quad e^x C_1' + (1+x) e^x C_2' = \dfrac{e^x}{x} \quad \Rightarrow \quad C_1' + (1+x) C_2' = \dfrac{1}{x}$

jelölés: $\quad e^x C_1' + x e^x C_2' = 0 \quad \Rightarrow \quad C_1' + x C_2' = 0$

$$\Rightarrow \quad \boxed{\begin{pmatrix} 1 & x \\ 1 & 1+x \end{pmatrix} \cdot \begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{x} \end{pmatrix}}$$

$\boxed{1p}$

① folyt

pl: Cramer-szabály $\det\begin{pmatrix}1 & x \\ 1 & 1+x\end{pmatrix} = 1+x-x=1$

$$c_1' = \frac{\det\begin{pmatrix}0 & x \\ \frac{1}{x} & 1+x\end{pmatrix}}{1} = \frac{-1}{1} = -1$$ 1p

$\hookrightarrow$ $c_1(x) = \int -1\,dx = -x$

$$c_2' = \frac{\det\begin{pmatrix}1 & 0 \\ 1 & \frac{1}{x}\end{pmatrix}}{1} = \frac{1}{x} \Rightarrow c_2(x) = \int \frac{1}{x}\,dx = \ln|x|$$

1p

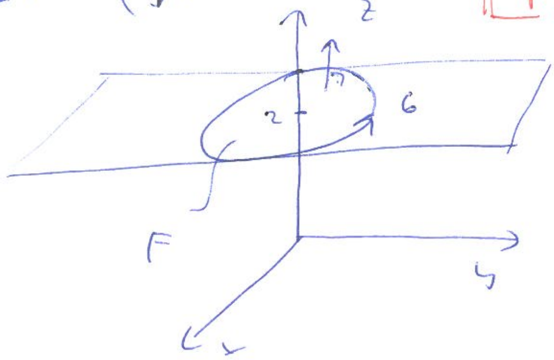
$\Rightarrow$ $y_{ip} = -x e^x + x\ln|x|\cdot e^x$

$$\boxed{y_{ált} = c_1 e^x + c_2 x e^x - x e^x + x\ln|x|\cdot e^x}$$ 1p

(2) $\underline{v}(x,y,z) = (2xy - z)\,\underline{i} + (x^2+z)\,\underline{j} + (y-x)\,\underline{k}$

$$G: \left.\begin{aligned} 16x^2+9y^2&=144 \\ z&=2 \end{aligned}\right\} \iff \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1$$

$\boxed{2p}$



$$\oint_G \underline{v}(\underline{r})\,d\underline{r} = ?$$

Pl: Stokes-tétellel

$$\oint_G \underline{v}(\underline{r})\,d\underline{r} = \iint_F \operatorname{rot}\underline{v}\,d\underline{F} \qquad F: \left(\frac{x}{3}\right)^2+\left(\frac{y}{4}\right)^2 \le 1$$

$\boxed{3p}$

$z = 2$

$d\underline{F} \parallel \underline{k}$

$$\operatorname{rot}\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy-z & x^2+z & y-x \end{vmatrix} = \underline{i}(1-1) - \underline{j}(-1-(-1)) + \underline{k}(2x-2x) = \underline{\underline{0}}$$

$\boxed{3p}$

$$\Rightarrow \oint_G \underline{v}(\underline{r})\,d\underline{r} = \iint_F \operatorname{rot}\underline{v}\,d\underline{F} = \underline{\underline{0}}$$

$\boxed{2p}$

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2. megoldás: görbementi integrállal: G paraméterezése:

$$\underline{r}(t) = (3\cos t, 4\sin t, 2) \qquad 0 \le t \le 2\pi$$

$$\dot{\underline{r}}(t) = (-3\sin t, 4\cos t, 0)$$

$\boxed{5p}$

$$\underline{v}(\underline{r}(t)) = (24\cos t\sin t - 2,\ 9\cos^2 t + 2,\ 4\sin t - 3\cos t)$$

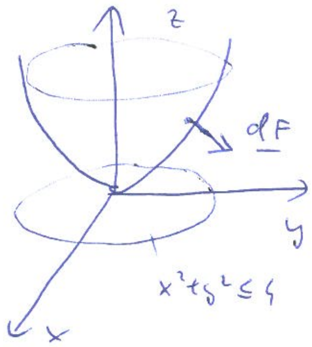
$$\Rightarrow \oint_G \underline{v}(\underline{r})\,d\underline{r} = \int_0^{2\pi} \underline{v}(\underline{r}(t))\cdot\dot{\underline{r}}(t)\,dt = \int_0^{2\pi} (-72\cos t\sin^2 t + 6\sin t + 36\cos^3 t + 8\cos t)\,dt$$

$$= \dots = \underline{\underline{0}}$$

$\boxed{5p}$

(3) $\underline{v}(\underline{r}) = \underline{r}$, $F: \left.\begin{array}{l} z = x^2+y^2 \\ x^2+y^2 \leq 4 \end{array}\right\}$

$d\underline{F} \cdot \underline{k} \leq 0$



F paraméterezése:

$$\boxed{\underline{r}(u,v) = (u\cos v, u\sin v, u^2)}$$

$$\boxed{\begin{array}{l} 0 \leq u \leq 2 \\ 0 \leq v \leq 2\pi \end{array}}$$

$\boxed{3p}$

$\hookrightarrow \underline{r}'_u = (\cos v, \sin v, 2u)$

$\underline{r}'_v = (-u\sin v, u\cos v, 0)$

$$\Rightarrow \underline{r}'_u \times \underline{r}'_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \underline{i}(-2u^2\cos v) - \underline{j}(2u^2\sin v) + \underline{k}(u\cos^2 v + u\sin^2 v) =$$

$$= (-2u^2\cos v, -2u^2\sin v, u)$$

$\boxed{2p}$

$\Rightarrow (\underline{r}'_u \times \underline{r}'_v) \cdot \underline{k} = u > 0 \rightsquigarrow$ meg kell fordítani az irányt!

$\boxed{1p}$

$\hookrightarrow \boxed{d\underline{F} = (2u^2\cos v, 2u^2\sin v, -u)\, du\, dv}$

$\underline{v}(\underline{r}(u,v)) = \underline{r}(u,v) = (u\cos v, u\sin v, u^2)$

$$\Rightarrow \iint_F \underline{v}(\underline{r})\, d\underline{F} = \int_0^{2\pi}\int_0^2 \underline{v}(\underline{r}(u,v)) \cdot (\underline{r}'_v \times \underline{r}'_u)\, du\, dv =$$

$$= \int_0^{2\pi}\int_0^2 (\underbrace{2u^3\cos^2 v + 2u^3\sin^2 v}_{2u^3} - u^3)\, du\, dv = \int_0^{2\pi}\underbrace{\int_0^2 u^3\, du}_{\left[\frac{u^4}{4}\right]_0^2 = 4}\, dv = 4 \cdot 2\pi =$$

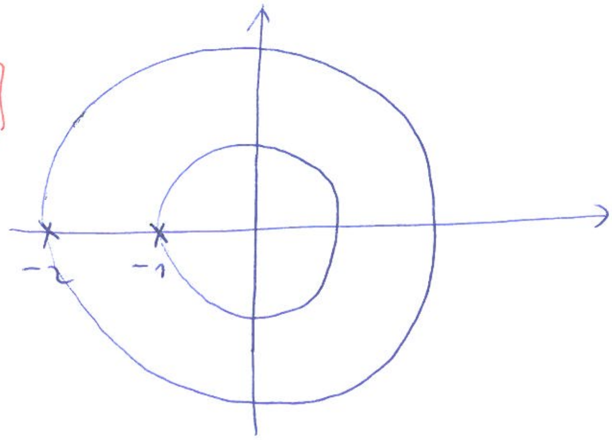
$$= \boxed{8\pi}$$

$\boxed{4p}$

(5)

$$f(z) = \frac{z}{(z+1)(z+2)} \qquad z_0 = 0 \text{ körüli Laurent-sora}$$

szinguláris helyek: $z = -1$
$z = -2$ $\boxed{1p}$



konvergencia tartományok:

T_1: $|z| < 1$

T_2: $1 < |z| < 2$ $\boxed{2p}$

T_3: $|z| > 2$

$$\frac{z}{(z+1)(z+2)} = \frac{A}{z+1} + \frac{B}{z+2} = \frac{A(z+2) + B(z+1)}{(z+1)(z+2)} \Longrightarrow \boxed{\begin{aligned} A &= -1 \\ B &= 2 \end{aligned}}$$

$$\hookrightarrow \quad f(z) = \frac{2}{z+2} - \frac{1}{z+1} \qquad \boxed{1p}$$

- T_1: $|z| < 1$ $\rightsquigarrow$ $\displaystyle \frac{1}{1+z} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n = 1 - z + z^2 - \ldots$

$$\rightsquigarrow \frac{2}{z+2} = \frac{1}{1+\frac{z}{2}} \underset{\left|\frac{z}{2}\right| < 1}{=} \sum_{n=0}^{\infty} \left(-\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n = 1 - \frac{1}{2} z + \frac{1}{4} z^2 - \ldots$$

$$\Longrightarrow \boxed{f(z) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2^n} - 1\right) z^n \quad \text{, ha } z \in T_1} \qquad \boxed{2p}$$

(5) folyt

- T_2: $1 < |z| < 2$

$$\rightsquigarrow \frac{2}{z+2} = \frac{1}{1+\frac{z}{2}} \underset{\left|\frac{z}{2}\right|<1}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n$$

$$\rightsquigarrow \frac{1}{z+1} = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}} \underset{\left|\frac{1}{z}\right|<1}{=} \frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} = \frac{1}{z} - \frac{1}{z^2} + \dots$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} z^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \quad , z \in T_2}$$

2p

- T_3: $|z| > 2$

$$\rightsquigarrow \frac{2}{z+2} = \frac{2}{z} \cdot \frac{1}{1+\frac{2}{z}} \underset{\left|\frac{2}{z}\right|<1}{=} \frac{2}{z} \sum_{n=0}^{\infty} \left(-\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{z^{n+1}} = \frac{2}{z} - \frac{4}{z^2} + \dots$$

$$\rightsquigarrow \frac{1}{z+1} = \frac{1}{z} \cdot \frac{1}{1+\frac{1}{z}} \underset{\left|\frac{1}{z}\right|<1}{=} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \quad , z \in T_3}$$

2p

(5) $\underline{v}(x,y,z) = (yz - xy)\underline{i} + (xz - \frac{x^2}{2} + yz^2)\underline{j} + (xy + y^2 z)\underline{k}$

$$\text{rot}\,\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz - xy & xz - \frac{x^2}{2} + yz^2 & xy + y^2 z \end{vmatrix} =$$

$$= \underline{i}\,(x + 2yz - (x + 2yz)) - \underline{j}\,(y - y) + \underline{k}\,(z - x - (z - x)) = 0$$

$\Rightarrow$ $\underline{v}$ mindenütt potenciálos $\Rightarrow$ $\exists\ u = u(x,y,z)$:

$\underline{v} = \text{grad}\, u$

3p

$$\left.\begin{aligned} &(1)\ \frac{\partial u}{\partial x} = yz - xy \\ &(2.)\ \frac{\partial u}{\partial y} = xz - \frac{x^2}{2} + yz^2 \\ &(3.)\ \frac{\partial u}{\partial z} = xy + y^2 z \end{aligned}\right\}$$

2p

$$\Rightarrow\ u(x,y,z) = \int (yz - xy)\,dx = xyz - \frac{x^2}{2}y + C(y,z)$$

$$\frac{\partial u}{\partial y} = xz - \frac{x^2}{2} + \underline{\frac{\partial C(y,z)}{\partial y}} \underset{(2.)}{=} xz - \frac{x^2}{2} + \underline{yz^2}$$

$$\Rightarrow\ C(y,z) = \int y z^2\,dy = \frac{y^2}{2} z^2 + C(z)$$

$$u(x,y,z) = xyz - \frac{x^2}{2}y + \frac{y^2}{2}z^2 + C(z)$$

$$\frac{\partial u}{\partial z} = xy + y^2 z + \frac{dC(z)}{dz} \underset{(3.)}{=} xy + y^2 z \Rightarrow \frac{dC(z)}{dz} = 0 \Rightarrow \underline{C(z) = C}$$

$$\Rightarrow\ \boxed{u(x,y,z) = xyz - \frac{x^2}{2}y + \frac{y^2}{2}z^2 + C}$$

5p