B variáns (–1–)

1, $y' + \frac{2y}{x^3} = x e^{\frac{1}{x^2}}$ $\int \frac{dy}{2y} = -\int \frac{dx}{x^3}$; $\ln|y| = +\frac{1}{x^2} + c$

$y_{h,\acute{a}}(x) = K \cdot e^{+\frac{1}{x^2}}$

$\underbrace{K' \cancel{e^{\frac{1}{x^2}}} + \cancel{\frac{-2}{x^3} K e^{\frac{1}{x^2}}}}_{y'} + \underbrace{\cancel{\frac{2K}{x^3} e^{\frac{1}{x^2}}}}_{\frac{2y}{x^3}} = x \cdot \cancel{e^{\frac{1}{x^2}}}$; $K' = x$; $K = \frac{x^2}{2}$

$y_{i,\acute{a}}(x) = \left(K + \frac{x^2}{2}\right) e^{\frac{1}{x^2}}$

— • —

2, ✓ 3, %

$\cancel{f(x) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \; ; \; x \in (-1,1)}$; ⊗

$\cancel{\frac{1}{1-2(x-3)-2} = \frac{1}{-2-(x-3)} = \frac{-1}{2} \cdot \frac{1}{1+\frac{x-3}{2}} = \frac{-1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k (x-3)^k}$

$\cancel{|x-3| < 2 \; ; \; -2 < x-3 < 2 \; ; \; x \in (1,5)}$ 5

— • —

4, $f(x,y) = \frac{-1}{y^4} - \frac{x^4}{2} + \frac{2x}{y^2}$; $y \neq 0$

$f'_x = -2x^3 + \frac{2}{y^2} = 0 \Rightarrow x^3 y^2 = 1$

$f'_y = 4y^{-5} - 4xy^{-3} = 0 \Rightarrow xy^2 = 1$

$(x^2 - 1)xy^2 = 0 \Rightarrow x = \pm 1$

$\Rightarrow \boxed{x = y = 1}$ vagy $\{x = +1, y = -1\}$

$f''_{xx} = -6x^2$; $f''_{xx}(1,1) = -6 < 0$

$f''_{yy} = -20y^{-6} + 12xy^{-4}$; $f''_{yy}(1,1) = -8$

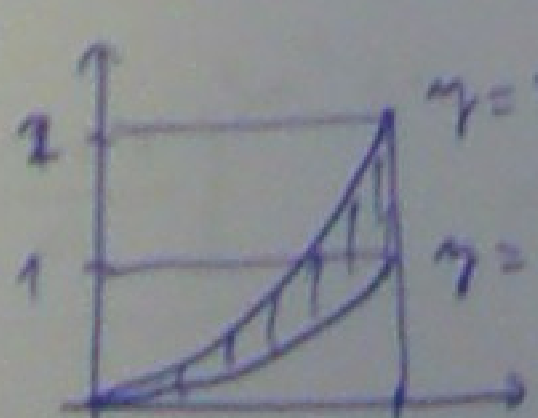
$f''_{xy} = -4y^{-3}$; $f''_{xy}(1,1) = -4$

$H(1,1) = \begin{vmatrix} -6 & -4 \\ -4 & -8 \end{vmatrix} = 48 - 16 = 32 > 0$

$\Rightarrow$ lokális max. van (1,1)-ben!

— • —

5, $\int_0^1 \int_{x^2}^{2x^2} f(x,y)\, dy\, dx = \int_{y=0}^{1} \int_{x=\sqrt{\frac{y}{2}}}^{\sqrt{y}} f(x,y)\, dx\, dy + \int_{y=1}^{2} \int_{x=\sqrt{\frac{y}{2}}}^{1} f(x,y)\, dx\, dy$



$y = 2x^2$; $x = \sqrt{\frac{y}{2}}$

$y = x^2$; $x = \sqrt{y}$

— • —

6, $\iint_T \frac{xy}{\sqrt{x^2+y^2}}\, dx\, dy = \int_{\varphi=\frac{3\pi}{2}}^{2\pi} \int_{r=3}^{4} \frac{r^2 \sin\varphi \cos\varphi}{\cancel{r}} \cancel{r}\, dr\, d\varphi$ (2) $= \frac{1}{2} \cdot \int_{\varphi=\frac{3\pi}{2}}^{2\pi} \sin(2\varphi)\, d\varphi \cdot \int_3^4 r^2\, dr =$

$= \frac{1}{2} \cdot \underbrace{\left[\frac{-\cos(2\varphi)}{2}\right]_{\frac{3\pi}{2}}^{2\pi}}_{-1}$ (2) $\cdot \underbrace{\left[\frac{r^3}{3}\right]_3^4}_{\frac{64-27}{3}}$ (2) $= \frac{-37}{6}$ (2)

$T: 9 \leq x^2 + y^2 \leq 16$;

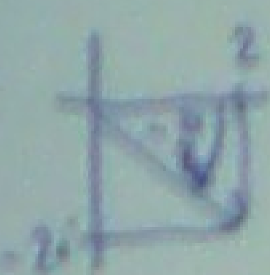
$x \geq 0$; $y \leq 0$

Brassán $\boxed{-2-}$

i, $f(x)=e^x=e^{3+x-3}=\underline{e^3\sum_{k=0}^{\infty}\frac{1}{k!}(x-3)^k}$; $\underline{K.T.=\mathbb{R}}$

ii, $g(x)=\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots=\underline{\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}}$; $\underline{K.T.=\mathbb{R}}$

7, $z_1=\ln(2-2i)=\underline{\ln(2\sqrt{2})-i\frac{\pi}{4}}$;



$z_2=\ln(e^2 i)=\underline{2+i\frac{\pi}{2}}$

$z_3=\ln(-e)=\underline{1-i\pi}$

8, $$I=\oint_{|z-i|=10}\frac{e^{3z}}{(z+2i-3)^2}\,dz=\frac{2\pi i}{1!}\cdot 3\cdot e^{3(3-2i)}=6\pi i e^9(\cos(-6)+i\sin(-6))=$$

$$=\underline{6\pi e^9(\sin 6+i\cos(6))}$$

9, $\operatorname{sh}(iz)=\dfrac{e^{iz}-e^{-iz}}{2}=i\,\dfrac{e^{iz}-e^{-iz}}{2i}=\underline{i\sin z}$; $\cos(iz)=\dfrac{e^{iiz}+e^{-iiz}}{2}=\dfrac{e^{-z}+e^{z}}{2}=\underline{\operatorname{ch} z}$

P1, $f(x,y)=2x^4y^2-x$; $P(-1,1)$

a, $f'_x(x,y)=\underline{8x^3y^2-1}$; $f'_x(-1,1)=-9$
$f'_y(x,y)=\underline{4x^4y}$; $f'_y(-1,1)=4$ $\Bigg\}$ $\underline{\operatorname{grad} f(P)=\begin{bmatrix}-9\\4\end{bmatrix}}$

b, $\underline{n}=\frac{v}{|v|}=\frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix}$; $\left.\frac{df}{d\underline{n}}\right|_P=\frac{1}{\sqrt{5}}\underbrace{(2\cdot(-9)+(-1)\cdot 4)}_{-22}=\underline{-\frac{22}{\sqrt{5}}}$

P2, $$\sum_{n=1}^{\infty}\underbrace{\left(\frac{n+5}{n}\right)^{3n^2+n}}_{a_n}x^n$$

$$\sqrt[n]{|a_n|}=\left(\frac{n+5}{n}\right)^{3n+1}=\underset{\downarrow\atop 1}{\frac{n+5}{n}}\cdot\underbrace{\left(1+\frac{5}{n}\right)^{3n}}_{\downarrow\atop e^{5\cdot 3}}\longrightarrow e^{15}\Longrightarrow\underline{R=e^{-15}}$$

1, $y' + \frac{y}{x^2} = x e^{\frac{1}{x}}$ $\qquad x \neq 0$

Inhomogén lineáris egyenlet.

$\boxed{6}$ Homogén rész: $\qquad y(x) \equiv 0$ megoldás

$$y' = \frac{dy}{dx} = -\frac{y}{x^2}\ ; \qquad \int \frac{dy}{y} = -\int \frac{dx}{x^2} \ ①$$

$$\ln|y| = \frac{1}{x} + c \ ② ; \qquad \underline{y_{h,á}(x) = K \cdot e^{\frac{1}{x}}} \ ① ; \quad K \in \mathbb{R}.$$

$\boxed{7}$ Inhomogén rész:

$$y_{i,p}(x) = K(x)\cdot e^{\frac{1}{x}} \ ① ; \quad y'_{i,p}(x) = K'(x) e^{\frac{1}{x}} - K(x)\cdot \frac{1}{x^2}\cdot e^{\frac{1}{x}} \ ②$$

$$\left(K' - \cancel{\frac{K}{x^2}}\right) \cancel{e^{\frac{1}{x}}} + \cancel{\frac{K}{x^2} e^{\frac{1}{x}}} = x \cancel{e^{\frac{1}{x}}} \ ② ; \quad K' = x ; \quad K(x) = \frac{x^2}{2} \ ①$$

$$\underline{y_{i,p}(x) = \frac{x^2}{2} e^{\frac{1}{x}}} \ ① ; \quad y_{i,á}(x) = y_{h,á}(x) + y_{i,p}(x) = \underline{\left(K + \frac{x^2}{2}\right) e^{\frac{1}{x}} ; \ K \in \mathbb{R}} \ ②$$

2, Hányados-kritériumból:

$\boxed{8}$ $\sum_{n=0}^{\infty} \underbrace{a_n x^n}_{b_n}$ abs. konvergens, ha $\exists \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1} x}{a_n}\right| < 1$

$$\iff |x| \cdot \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 \implies \underline{|x| < \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = R}$$

<u>VAGY!</u> $\qquad$ (Feltéve, hogy létezik a limesz, és nem 0.)

Gyökkritériumból:

$\boxed{8}$ $\sum_{n=0}^{\infty} \underbrace{a_n x^n}_{b_n}$ abs. konv. $\iff \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n x^n|} = \overline{\lim_{n\to\infty}} |x| \cdot \sqrt[n]{|a_n|} < 1$

$$\implies \underline{|x| < \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|}}} = R. \qquad (\text{Ha a } \overline{\lim} \text{ nem } 0.)$$

-2-

i, $f(x) = e^x$ Tudjuk, hogy $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (1) $\forall x \in \mathbb{R}$ esetén.

ii, $x_0 = 5$

$$f(x) = e^{5+x-5} = e^5 \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(x-5)^k \quad (2) \quad ; \quad K.T. = \mathbb{R} \quad (1)$$

~~ii, $x_0 = 5$ $f(x) = e^{5+x-5} = e^5 \cdot \sum_{k=0}^{\infty} \frac{1}{k!}(x-5)^k$ (2) ; K.T. = $\mathbb{R}$ (1)~~

b, $f(x,y) = \frac{1}{x^3} + y^3 - \frac{3y}{x}$; $x \neq 0$ esetén diff.-ható.

$$f'_x(x,y) = -3x^{-4} + 3yx^{-2} \quad (2) \quad ; \quad f'_y(x,y) = 3y^2 - \frac{3}{x} \quad (2)$$

$$f'_x = 0 \; ; \Rightarrow 1 - x^2 y = 0 \quad (1) \quad ; \quad f'_y = 0 \Rightarrow xy^2 - 1 = 0 \quad (1)$$

$$\left.\begin{array}{l} x^2 y = 1 \qquad xy^2 = 1 \\ xy(x-y) = 0 \end{array}\right\} \Rightarrow x = y = 1$$

A függvénynek csak az $x = y = 1$ (1) helyen lehet lokális szélső-értéke.

$$f''_{xx} = 12x^{-5} - 6yx^{-3} \quad ; \quad f''_{xx}(1,1) = 12 - 6 = 6 \quad (2) \quad > 0$$

$$f''_{yy} = 6y \quad ; \quad f''_{yy}(1,1) = 6 \quad (2)$$

$$f''_{xy} = 3 \cdot x^{-2} \quad ; \quad f''_{xy}(1,1) = 3 \quad (2)$$

$$H(1,1) = \begin{vmatrix} 6 & 3 \\ 3 & 6 \end{vmatrix} = 36 - 9 = 27 > 0 \quad (2) \; ; \longrightarrow$$

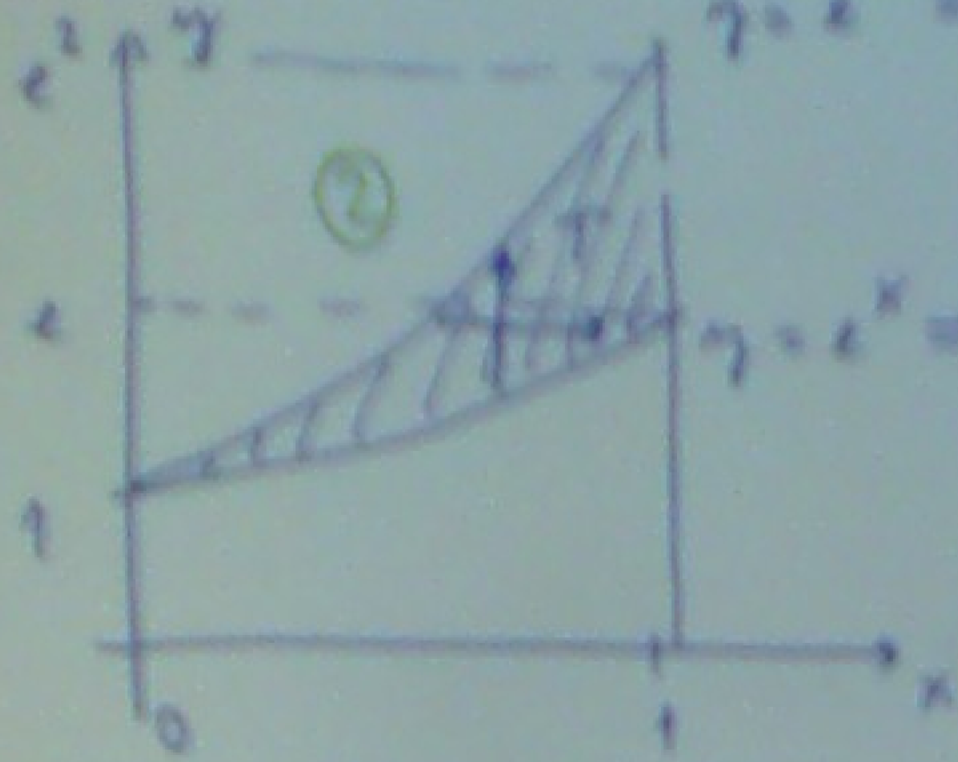
lokális minimuma van f-nek az (1,1) pontban! (2)

3, ii, $g(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ (4)

K.T.: $\mathbb{R}$ (1)

5, $\int_0^1 \int_{e^x}^{e^{2x}} f(x,y)\,dy\,dx = \int_{y=1}^{e}\int_{x=\frac{1}{2}\ln y}^{\ln y} f(x,y)\,dx\,dy + \int_{y=e}^{e^2}\int_{x=\frac{1}{2}\ln y}^{1} f(x,y)\,dx\,dy$ (2)

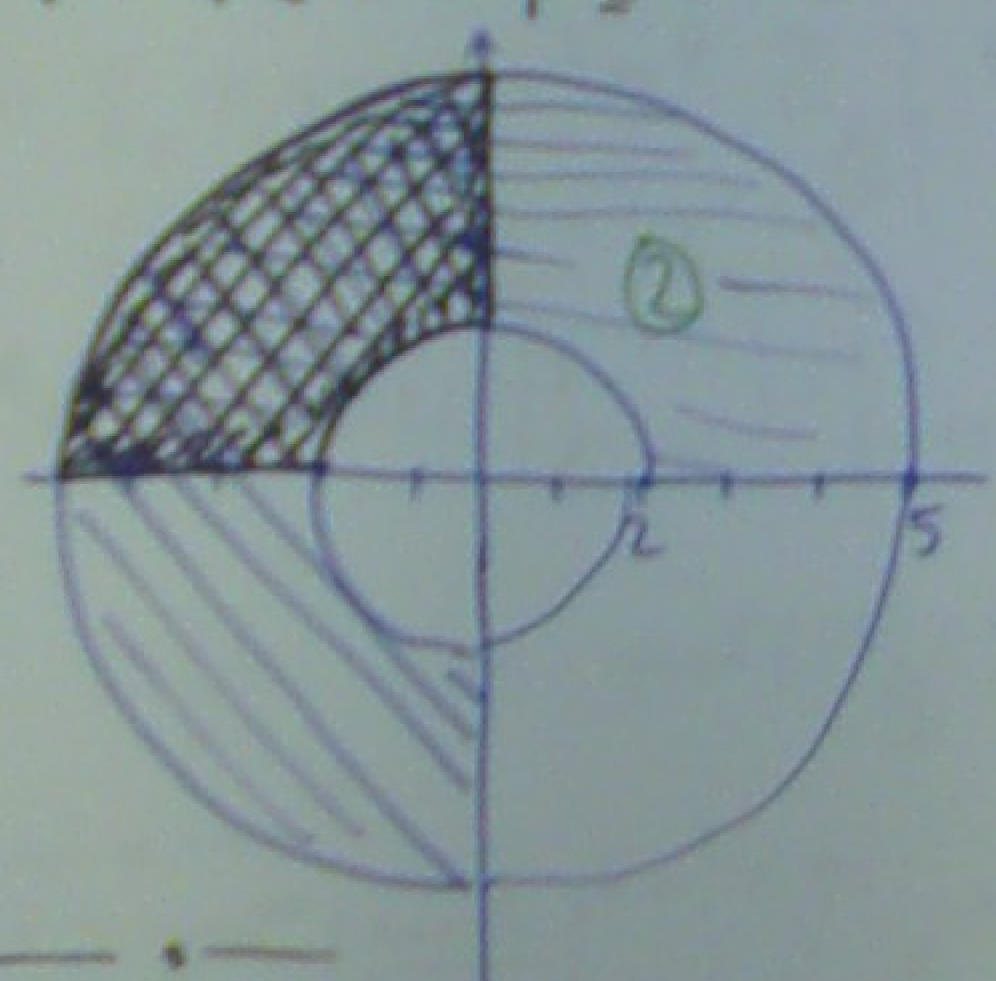
$y=e^{2x} \Rightarrow x=\frac{1}{2}\ln y$ (2)



$y=e^{x} \Rightarrow x=\ln y$ (2)

6, $\iint_T \frac{xy}{\sqrt{x^2+y^2}}\,dx\,dy =$ síkbeli polár $\int_{\varphi=\frac{\pi}{2}}^{\pi}\int_{r=2}^{5} \frac{r^2\sin\varphi\cos\varphi}{\cancel{r}}\cdot\cancel{r}\,dr\,d\varphi =$ (2)

T: $4\le x^2+y^2\le 25;\ x\le 0;\ y\ge 0$



$x=r\cos\varphi$

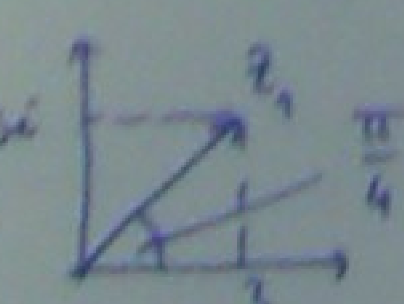
$y=r\sin\varphi$

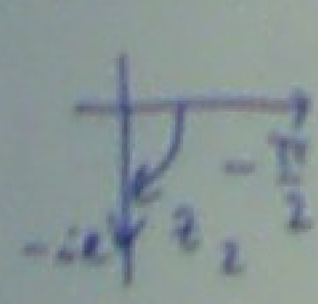
$|J|=r$

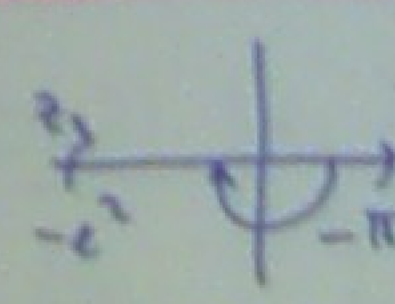
$= \underbrace{\int_{\frac{\pi}{2}}^{\pi}\frac{\sin(2\varphi)}{2}\,d\varphi}_{\frac{1}{2}\cdot\left[\frac{-\cos(2\varphi)}{2}\right]_{\frac{\pi}{2}}^{\pi}} \underbrace{\int_{r=2}^{5} r^2\,dr}_{\left[\frac{r^3}{3}\right]_2^5} =$ (2) (2)

$= \frac{1}{12}\cdot(-1-(+1))\cdot(125-8) = \frac{-117}{6}$ (2)

7, $z_1=\ln(3+3i)=\ln\sqrt{9+9}+i\cdot\frac{\pi}{4}=\ln(3\sqrt{2})+i\frac{\pi}{4}$ (4)







$z_2=\ln(-ie)=\ln e+i\cdot\left(-\frac{\pi}{2}\right)=1-i\frac{\pi}{2}$ (3)

$z_3=\ln(-e^2)=\ln e^2+i(-\pi)=2-i\pi$ (3)

e^{4z} ← reguláris $\mathbb{C}$-n $\lfloor - 4 - \rfloor$

$$\oint \frac{e^{4z}}{(z-3i+2)^2}\,dz = \frac{2\pi i}{1!}\cdot f'(z_0) = 2\pi i\cdot 4\cdot e^{4(-2+3i)} \; (2) = 8\pi i\, e^{-8}(\cos 12 + i \sin 12)$$

$|z-i| = 100$

$$= \cancel{8\pi i e^{-8} \ldots} = \underline{8\pi e^{-8}(-\sin 12 + i\cos 12)} \; (2)$$

(2) $\left\{ \begin{array}{l} z_0 = -2+3i \\ f(z) = e^{4z} \text{ reg. } \mathbb{C}\text{-n} \\ \gamma \text{ - körülveszi } z_0\text{-t} \\ n = 2 \end{array} \right\}$ C.-II. int. formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz \; (2)$$

— • —

9, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ (2) ; $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ (1) ; $\operatorname{sh} z = \dfrac{e^{z} - e^{-z}}{2}$ (2) ; $\operatorname{ch} z = \dfrac{e^{z} + e^{-z}}{2}$ (2)

Kapcsolatok: $\sin(iz) = \dfrac{e^{-z} - e^{z}}{2i} = i \operatorname{sh} z$ (2) ; ~~...~~ ~~(2)~~

~~$\cos(iz) = \frac{e^{-z} + e^{z}}{2} = \ldots$~~ (2) ; $\operatorname{ch}(iz) = \cos z$

→ $\operatorname{ch}(iz) = \dfrac{e^{iz} + e^{-iz}}{2} = \cos z$

— • —

P1: $f(x,y) = 3x^3y^4 + y$; $P(1,-1)$

a, $\left.\begin{array}{l} f'_x(x,y) = \underline{9x^2y^4} \; (2) \\ f'_y(x,y) = \underline{12x^3y^3 + 1} \; (2) \end{array}\right\}$ $\mathbb{R}^2$-en folytonosak $\Rightarrow \exists \operatorname{grad} f(P)$ (2)

b, $\operatorname{grad} f(P) = \begin{bmatrix} f'_x(P) \\ f'_y(P) \end{bmatrix} = \underline{\begin{bmatrix} 9 \\ -11 \end{bmatrix}}$ (1) ; $|\underline{v}| = \sqrt{1+4} = \sqrt{5}$ (1) ; $\underline{n} := \dfrac{\underline{v}}{|\underline{v}|}$

$$\left.\frac{df}{d\underline{n}}\right|_P = \underline{n}\cdot \operatorname{grad} f(P) \; (2) = \frac{1}{\sqrt{5}}\cdot \underbrace{(9\cdot(-1) + (-11)\cdot 2)}_{-31} = \underline{\underline{-\frac{31}{\sqrt{5}}}} \; (2)$$

— • —

P2: $$\sum_{n=1}^{\infty} \underbrace{\left(\frac{n+3}{n}\right)^{4n^2+n}}_{a_n} x^n$$

Gyökkrit.: $\sqrt[n]{|a_n|} = \left(\dfrac{n+3}{n}\right)^{4n+1}$ (2) $= \underbrace{\left(\dfrac{n+3}{n}\right)}_{\downarrow\, 1 \; (1)} \cdot \underbrace{\left(1+\dfrac{3}{n}\right)^{4n}}_{\downarrow\, e^{3\cdot 4} \; (2)}$ (2) $\longrightarrow e^{12}$

$$R = \frac{1}{e^{12}} = \underline{\underline{e^{-12}}} \; (1)$$