MINTAMEGOLDÁS E1-1 Inf. an. 2. VD2 2014.06.05.

1, $y' + \frac{y}{x} = e^x + \frac{e^{3x}}{x}$

$\boxed{13}$

Homogén: $y' + \frac{y}{x} = 0 \Rightarrow y(x) \equiv 0$, vagy $\int \frac{dy}{y} = -\int \frac{dx}{x}$

$\ln y = -\ln x + C \Rightarrow \boxed{y_{H,\text{ált}}(x) = K \cdot \frac{1}{x} \,;\, K \in \mathbb{R}}$ 5

Inhomogén:

$y_{I,p}(x) = K(x) \cdot \frac{1}{x} \,;\, y'_{I,p}(x) = K'(x) \cdot \frac{1}{x} - K(x) \cdot \frac{1}{x^2}$ 2

Beírva:

$\frac{K'(x)}{x} - \cancel{\frac{K(x)}{x^2}} + \cancel{\frac{K(x)}{x \cdot x}} = e^x + \frac{e^{3x}}{x} \,;\, K'(x) = xe^x + e^{3x}$ 2

$K(x) = \int (\underset{u}{x}\, \underset{v'}{e^x} + e^{3x})\, dx = xe^x - \int 1 \cdot e^x\, dx + \frac{1}{3}e^{3x} = (x-1)e^x + \frac{e^{3x}}{3}$

$y_{I,p}(x) = \left((x-1)e^x + \frac{e^{3x}}{3}\right) \cdot \frac{1}{x}$ 2

$y_{I,\text{ált}}(x) = y_{I,p}(x) + y_{H,\text{ált}}(x) = \underline{\underline{\frac{K}{x} + \frac{x-1}{x}e^x + \frac{e^{3x}}{3x}}} \,;\, K \in \mathbb{R}$ 2

2a, $\varphi_1(x) = x \sin(2x) \,; \Rightarrow \lambda_{1,2,3,4} = \pm 2i$ 2-szeres gyökpár

$\boxed{7}$

$\varphi_2(x) = e^{3x} \Rightarrow \lambda_5 = 3$ 1-szeres gyök

2

Kar. egyenlet:

$(\lambda + 2i)^2 (\lambda - 2i)^2 (\lambda - 3) = (\lambda^2 + 4)^2 (\lambda - 3) =$ 2

$= (\lambda^4 + 8\lambda^2 + 16)(\lambda - 3) = \lambda^5 - 3\lambda^4 + 8\lambda^3 - 24\lambda^2 + 16\lambda - 48$

$\Rightarrow \underline{y^{(5)} - 3y^{(4)} + 8y^{(3)} - 24y'' + 16y' - 48y = 0}$ 1

$\underline{y_{H,\text{ált}}(x) = (Ax+B)\sin(2x) + (Cx+D)\cos(2x) + Ee^{3x}}$, 2

$\underline{A, B, C, D, E \in \mathbb{R}}$.

2, b, $y'' + y = \sin x$

$\boxed{4}$ $\lambda^2 + 1 = (\lambda + i)(\lambda - i) = 0 \Rightarrow \lambda_{1,2} = \pm i$ Külső rezonancia!

$\Rightarrow y_{i,p}(x) = x(A \sin x + B \cos x)$

3, a, $\boxed{4}$

$f(x) = \operatorname{ch}(3x^2) = \sum_{n=0}^{\infty} \frac{(3x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{3^{2n}}{(2n)!} x^{4n}$ (2) ; $R = \infty$ (1)

$a_4 = \frac{3^2}{2!} = \frac{9}{2}$ (1)

$4n = 4,\ n = 1$

b, $\boxed{8}$

$g(x) = \frac{1}{\sqrt[3]{8 + 5x^2}} = \left(8\left(1 + \frac{5x^2}{8}\right)\right)^{-1/3} = \frac{1}{2}\left(1 + \frac{5x^2}{8}\right)^{-1/3} =$

$= \frac{1}{2} \sum_{n=0}^{\infty} \binom{-1/3}{n} \left(\frac{5}{8}\right)^n \cdot x^{2n}$ (4) ; $\left|\frac{5x^2}{8}\right| < 1 \Rightarrow |x| < \sqrt{\frac{8}{5}} = R$ (2)

$a_4 = \frac{1}{2} \cdot \binom{-1/3}{2} \left(\frac{5}{8}\right)^2 = \frac{1}{2} \cdot \frac{(-1/3)\cdot(-4/3)}{2} \cdot \frac{25}{64}$ (2)

$2n = 4$

$n = 2$

5, a, D.: Az $f(x) = \sum_{n=0}^{\infty} f_n(x)$ függvénysor az I intervallumon

$\boxed{3}$ egyenletesen konvergens, ha $\forall \varepsilon > 0$ esetén $\exists N(\varepsilon) \in \mathbb{N}$, hogy $\forall n > N(\varepsilon)$, $\forall x \in I$ esetén $\left| f(x) - \sum_{k=0}^{n} f_k(x) \right| < \varepsilon$.

b, T.: (Weierstrass kritérium) Ha $\forall n \in \mathbb{N}$ és $\forall x \in I$ esetén

$\boxed{3}$ $|f_n(x)| \le b_n$, és $\sum_{n=0}^{\infty} b_n < \infty$, akkor a $\sum_n f_n$ függvénysor egyenletesen konvergens I-n. (3) (És abszolút konvergens I minden pontjában.)

5/c, A sorokra vonatkozó Cauchy-kritériumot használjuk:

[6] $\sum_{n\in\mathbb{N}} b_n < \infty \Longrightarrow (\forall \varepsilon>0)(\exists N(\varepsilon)): \forall n>m>N(\varepsilon)$ esetén

$$\left|\sum_{k=m}^{n} b_k\right| = \sum_{k=m}^{n} b_k < \varepsilon.$$

Tehát $\forall x \in I$-re

$$\left|\sum_{k=m}^{n} f_k(x)\right| \le \sum_{k=m}^{n} |f_k(x)| \le \sum_{k=m}^{n} b_k < \varepsilon, \text{ ha } m,n>N(\varepsilon)$$ ✓

4, a, $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = A \in \mathbb{R}$, ha

[3] i, (x_0,y_0) torlódási pontja D_f-nek, és

ii, $\forall \varepsilon>0$ esetén $\exists \delta(\varepsilon)>0$, hogy

$$|f(x,y)-A|<\varepsilon, \text{ ha } (x,y)\in \dot{K}_{\delta(\varepsilon)}(x_0,y_0)\cap D_f$$

$$\left[\begin{array}{l} \|(x-x_0, y-y_0)\| = \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta(\varepsilon), \\ (x,y)\neq(x_0,y_0), \quad (x,y)\in D_f \end{array}\right]$$

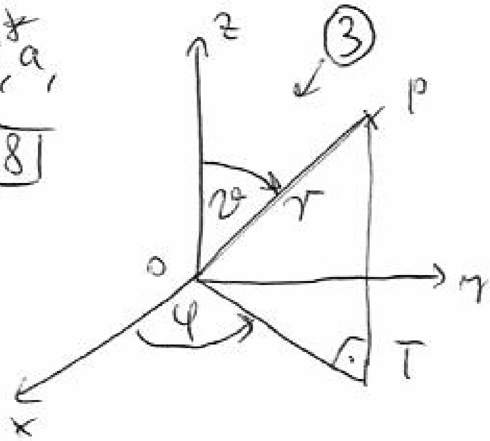
b, i, $x = r\cos\varphi$, $y = r\sin\varphi$

[4] $$f(x,y) = \frac{x^2y}{x^2+y^2} = \frac{r^3\cos^2\varphi\sin\varphi}{r^2(\cos^2\varphi+\sin^2\varphi)} = \underset{\downarrow\atop 0}{r}\underbrace{\cos^2\varphi\sin\varphi}_{\text{korlátos}} \xrightarrow[r\to 0]{} \underline{\underline{0}}$$

ii,

[5] $$g(x,y) = \frac{\sin(x^2y)}{x^2+y^2} = \underbrace{\frac{\sin(x^2y)}{x^2y}}_{\substack{\downarrow (x,y)\to(0,0)\\ 1}} \cdot \underbrace{\frac{x^2y}{x^2+y^2}}_{\substack{\downarrow (x,y)\to(0,0)\\ 0}} = 1\cdot 0 = 0$$

6, a, $\boxed{8}$



$$\left.\begin{aligned} x &= r\sin\vartheta\cos\varphi \\ y &= r\sin\vartheta\sin\varphi \\ z &= r\cos\vartheta \end{aligned}\right\}$$

$$J = \begin{vmatrix} x'_r & x'_\vartheta & x'_\varphi \\ y'_r & y'_\vartheta & y'_\varphi \\ z'_r & z'_\vartheta & z'_\varphi \end{vmatrix} \text{①} = \begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix} =$$

$$= \cos\vartheta \underbrace{\begin{vmatrix} r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \end{vmatrix}}_{r^2\cos\vartheta\sin\vartheta\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1}} + r\sin\vartheta \underbrace{\begin{vmatrix} \sin\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \end{vmatrix}}_{r\sin^2\vartheta\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{1}} =$$

alsó sor szerint

$$= r^2\sin\vartheta(\cos^2\vartheta+\sin^2\vartheta) = \underline{\underline{r^2\sin\vartheta}} \text{ ④}$$

③ $r^2\sin\vartheta\, d\varphi\, d\vartheta\, dr$ ✓

$\boxed{6}$ b,
$$\iiint_V x^3 y z\, dV = \int_{r=0}^{2}\int_{\vartheta=0}^{\pi/2}\int_{\varphi=0}^{\pi/2} (r\sin\vartheta\cos\varphi)^3(r\sin\vartheta\sin\varphi)(r\cos\vartheta) \checkmark =$$

$$= \left(\int_{r=0}^{2} r^7\, dr\right)\cdot\left(\int_{\vartheta=0}^{\pi/2}\sin^5\vartheta\cos\vartheta\, d\vartheta\right)\cdot\left(\int_{\varphi=0}^{\pi/2}\cos^3\varphi\sin\varphi\, d\varphi\right) =$$

$$= \left[\frac{r^8}{8}\right]_0^2 \cdot \left[\frac{\sin^6\vartheta}{6}\right]_{\vartheta=0}^{\pi/2} \cdot \left[-\frac{\cos^4\varphi}{4}\right]_{\varphi=0}^{\pi/2} =$$

$$= \frac{2^8}{8}\cdot\frac{1}{6}\cdot\frac{1}{4} = \frac{4}{3} \text{ ③}$$

7*

$u(x,y) = \alpha x^3 - 3xy^2 - 2xy$

[3] a, $u'_x = 3\alpha x^2 - 3y^2 - 2y \; ; \; u'_y = -6xy - 2x$

$\Delta u = 6\alpha x - 6x = 0 \implies \boxed{\alpha = +1}$

[4] b, $f'(1-2i) = u'_x(1,-2) + i \underbrace{v'_x(1,-2)}_{= -u'_y(1,-2)} = +3 - 12 + 4 - i(12-2) =$

$= \underline{\underline{-5 - 10i}}$

[6] c, $v'_y(x,y) = u'_x = +3x^2 - 3y^2 - 2y$

$\implies v(x,y) = \int v'_y \, dy = +3x^2 y - y^3 - y^2 + C(x)$ (3)

$v'_x = +6xy + C'(x) \overset{!}{=} -u'_y = +6xy + 2x \implies C'(x) = 2x;$

$C(x) = x^2 + K$ (2)

$\underline{\underline{v(x,y) = 3x^2 y - y^3 - y^2 + x^2 + K}}$ (1) ; $K \in \mathbb{R}$

8* a, T.: (Cauchy alaptétel) Ha f reguláris a $T \subset \mathbb{C}$

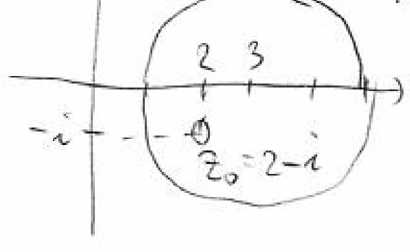
[3] egyszeresen összefüggő tartományon, és $L \subset T$ zárt görbe, akkor $\oint_L f(z)\,dz = 0$

[10] b, (2)

$$\oint_{|z-3|=2} \frac{\sin(2z)}{(z-2+i)^4}\,dz = \frac{2\pi i}{3!} \left.\frac{d^3 \sin(2z)}{dz^3}\right|_{z=2-i} = \quad (3)$$

$n = 3$

$z_0 = 2 - i$



$$= \frac{2\pi i}{6} \cdot 2^3 \cdot \Big(-\underbrace{\cos(4-2i)}_{\cos 4 \cos(2i) + \sin 4 \sin(2i)}\Big) \overset{(2)}{=} \underline{\underline{\frac{-8\pi}{3}\left(i \cos 4 \,\mathrm{ch}\, 2 - \sin 4 \,\mathrm{sh}\, 2\right)}} \quad (3)$$