1, i, $\boxed{7}$

$$I = \int \frac{1}{e^x+2}\,dx \underset{\substack{u=e^x \\ x=\ln u,\; dx=\frac{du}{u}}}{\Longrightarrow} \int \frac{1}{u+2}\cdot\frac{du}{u} = \quad (5)$$

$$\frac{1}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2} \quad (3) \quad \Rightarrow \quad 1 = A(u+2) + Bu$$

$$u=0:\; 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$u=-2:\; 1 = -2B \Rightarrow B = -\frac{1}{2} \qquad (4)$$

$$= \frac{1}{2}\int \frac{1}{u}\,du - \frac{1}{2}\int \frac{1}{u+2}\,du = \frac{1}{2}\ln|u| - \frac{1}{2}\ln|u+2| + C \quad (3)$$

$$I = \frac{1}{2}\ln \underbrace{e^x}_{>0} - \frac{1}{2}\ln(e^x+2) + C = \underline{\underline{\frac{x}{2} - \frac{1}{2}\ln(e^x+2) + C}} \quad (2)$$

---

ii, $\boxed{8}$

$$\int \frac{(x-2)^2}{x^2+2^2}\,dx = \int \frac{x^2-4x+4}{x^2+4}\,dx = \int\left(1 - \frac{4x}{x^2+4}\right)dx =$$

$$= \int 1\,dx - 2\int \frac{2x}{x^2+4}\,dx \;(4) = \underline{\underline{x \;(1) - 2\ln(x^2+4) \;(3) + C}}$$

$\frac{f'}{f}$ alak

2, $\boxed{10}$

$$\int_1^\infty f(x)\,dx = \lim_{\Omega\to\infty} \int_1^\Omega f(x)\,dx \quad (3)$$

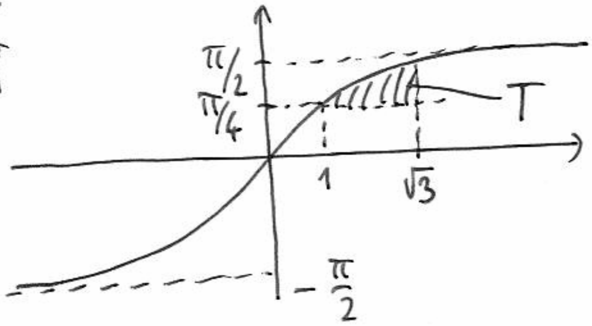
$(f: \mathbb{R}^+ \to \mathbb{R}$ folytonos, így $\forall \Omega \geq 1$ esetén $f \in R[1,\Omega])$

$$\int_1^\infty \frac{d}{dx}\left(x \sin\frac{\pi}{x}\right)dx = \lim_{\Omega\to\infty} \int_1^\Omega \frac{d}{dx}\left(x\sin\frac{\pi}{x}\right)dx = \lim_{\Omega\to\infty}\left[x\sin\frac{\pi}{x}\right]_1^\Omega \;(2) =$$

$$= \lim_{\Omega\to\infty}\left(\Omega\sin\frac{\pi}{\Omega} - 1\cdot\sin\left(\frac{\pi}{1}\right)\right) = \lim_{\substack{u\to 0+ \\ (u=\frac{1}{\Omega})}} \frac{\sin(\pi u)}{u} \;(2) - 0 \;(1) =$$

$$= \lim_{u\to 0+} \pi \underbrace{\frac{\sin(\pi u)}{\pi u}}_{\downarrow u\to 0 \atop 1} = \underline{\underline{\pi}} \quad (2)$$

3/ $\boxed{15}$



$y = \operatorname{arctg} x$ (6)

$$T = \int_{\underbrace{x=\operatorname{tg}\frac{\pi}{4}}_{1}}^{\sqrt{3}} \left(\operatorname{arctg} x - \frac{\pi}{4}\right) dx =$$

$$= \int_1^{\sqrt{3}} 1 \cdot \operatorname{arctg} x \, dx - \int_1^{\sqrt{3}} \frac{\pi}{4} \, dx = \overbrace{\Big[x \operatorname{arctg} x\Big]_1^{\sqrt{3}} - \int_1^{\sqrt{3}} \frac{x}{1+x^2} \, dx}^{(3)} - \frac{\pi}{4}(\sqrt{3}-1) =$$

f'/f alak (1)

$u' = 1$; $v = \operatorname{arctg} x$
$u = x$; $v' = \frac{1}{1+x^2}$

$$= \sqrt{3} \operatorname{arctg} \sqrt{3} - 1 \cdot \operatorname{arctg} 1 - \frac{1}{2}\Big[\ln(1+x^2)\Big]_1^{\sqrt{3}} - \frac{\pi}{4}(\sqrt{3}-1) =$$ (3)

(2)

$$= \sqrt{3}\,\frac{\pi}{3} - \cancel{\frac{\pi}{4}} \underbrace{- \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2}_{-\ln\sqrt{2}} - \frac{\pi}{4}\sqrt{3} + \cancel{\frac{\pi}{4}} = \sqrt{3}\,\pi \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{\frac{1}{12}} - \ln\sqrt{2}$$

4/ $\boxed{12}$ <u>T.:</u> Ha $\lim_{n\to\infty} a_n = A$ és $\lim_{n\to\infty} a_n = B$, akkor $A = B$. (3)

<u>B.:</u> Indirekt. T.f.h. $A \neq B$, és $\varepsilon := \frac{|A-B|}{3} > 0$

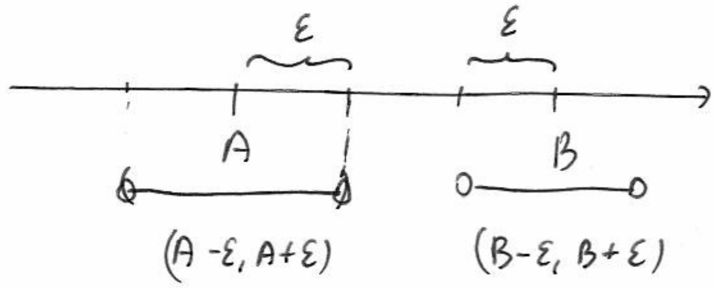
$\lim_{n\to\infty} a_n = A \implies \exists N_A \in \mathbb{N}; \ \forall n > N_A: \quad a_n \in (A-\varepsilon, A+\varepsilon)$

$\lim_{n\to\infty} a_n = B \implies \exists N_B \in \mathbb{N}: \ \forall n > N_B: \quad a_n \in (B-\varepsilon, B+\varepsilon)$

$N := \max(N_A, N_B)$

$\forall n > N: \ a_n \in (A-\varepsilon, A+\varepsilon) \cap (B-\varepsilon, B+\varepsilon) = \emptyset$ ↯ ✓

(9)



5, $\boxed{28}$

$f(x) = x \ln^2(-x)$; $D_f = (-\infty, 0)$ ①; $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \overset{-\infty}{x} \overset{\infty}{\ln^2(-x)} = -\infty$ ②

⑤ $\lim\limits_{x\to 0-} x\ln^2(-x) = \lim\limits_{x\to 0-} \dfrac{\ln^2(-x)}{(1/x)} \overset{\text{"}\frac{\infty}{\infty}\text{"}}{\underset{L'H}{=}} \lim\limits_{x\to 0-} \dfrac{2\ln(-x)\cdot\left(\frac{-1}{-x}\right)}{\left(-\frac{1}{x^2}\right)} = \lim\limits_{x\to 0-} \dfrac{-2\ln(-x)}{\left(\frac{1}{x}\right)} =$

$\overset{\text{"}\frac{\infty}{\infty}\text{"}}{\underset{L'H}{=}} \lim\limits_{x\to 0-} \dfrac{\left(\frac{-2}{x}\right)}{\left(\frac{-1}{x^2}\right)} = \lim\limits_{x\to 0-} 2x = 0$

$\boxed{9}$ $f'(x) = \ln^2(-x) + x\, 2\ln(-x)\cdot\left(\frac{-1}{-x}\right)$

$= \ln(-x)\left(\ln(-x)+2\right)$ ③

$f'(x) = 0 \iff x = -1$ vagy $x = -e^{-2}$ ③

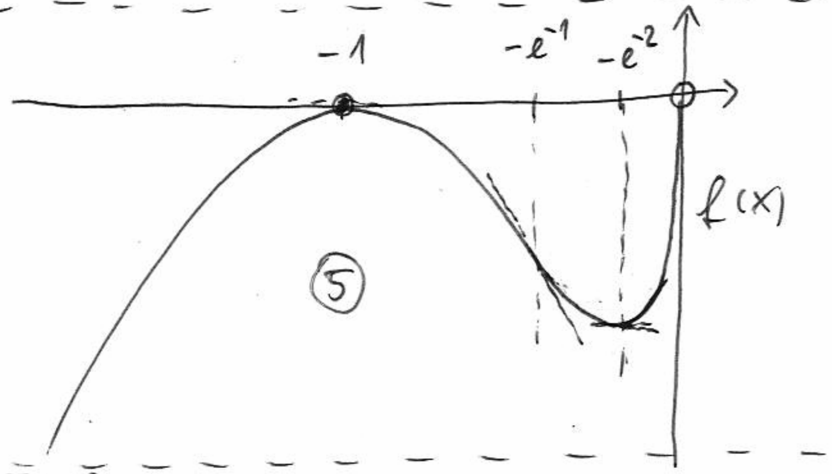
③

x	$(-\infty,-1)$	-1	$(-1,-e^{-2})$	$-e^{-2}$	$(-e^{-2},0)$
f'	+	0	–	0	+
f	↗	lok. max.	↘	lok. min.	↗

$\boxed{6}$ $f''(x) = \left(\frac{-1}{-x}\right)\left(\ln(-x)+2\right) + \ln(-x)\cdot\left(\frac{-1}{-x}\right) = \dfrac{2(1+\ln(-x))}{x}$ ②; $f''(x) = 0 \iff x = -e^{-1}$ ②

x	$(-\infty,-e^{-1})$	$-e^{-1}$	$(-e^{-1},0)$
f''	–	0	+
f	∩	infl. pont	∪

②



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6, $\boxed{10}$

Ha $x > 0$, akkor $\operatorname{arctg}(x+1) \in \left(0, \frac{\pi}{2}\right)$ ②

Legyen $f(x) = 2\sin(\pi x) - \operatorname{arctg}(x+1)$

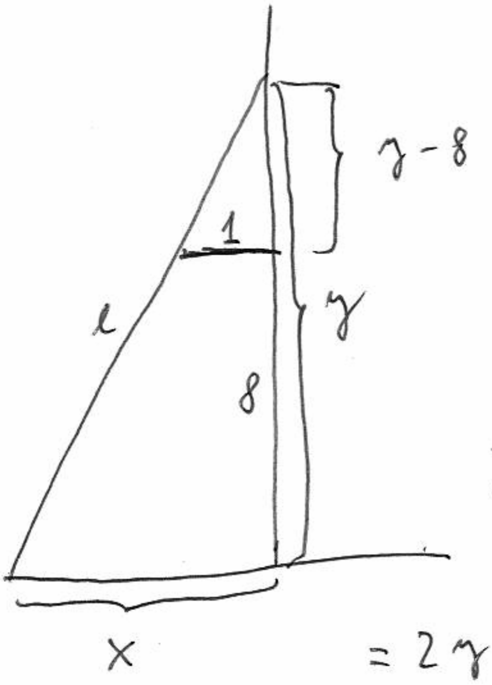
$f(2n) = 2\underbrace{\sin(2\pi n)}_{0} - \operatorname{arctg}(2n+1) \in \left(-\frac{\pi}{2}, 0\right)$

$f(2n+\frac{1}{2}) = 2\underbrace{\sin\left(2\pi n + \frac{\pi}{2}\right)}_{1} - \operatorname{arctg}\left(2n+\frac{3}{2}\right) \in \left(2-\frac{\pi}{2}, 2\right)$

Tehát $f(2n) < 0$ ②, $f(2n+\frac{1}{2}) > 0$ ②. f a $[2n, 2n+\frac{1}{2}]$ intervallumon folytonos, így a Bolzano-tétel értelmében $\exists\, \xi \in (2n, 2n+\frac{1}{2})$, melyre $f(\xi) = 0$, tehát a vizsgált egyenletnek ξ gyöke. ④

IMSC 16

$$\frac{y}{x} = \frac{y-8}{1} \Rightarrow x = \frac{y}{y-8}$$



$$l(y) = \sqrt{x^2 + y^2} = \sqrt{\frac{y^2}{(y-8)^2} + y^2} \quad (y > 8)$$

l minimális $\Leftrightarrow$ l^2 minimális

$$\frac{d}{dy}\left(l^2(y)\right) = \left(y^2\left(1 + \frac{1}{(y-8)^2}\right)\right)' =$$

$$= 2y\left(1 + \frac{1}{(y-8)^2}\right) - \frac{2y^2}{(y-8)^3} = \frac{2y\left((y-8)^3 + \cancel{y} - 8 - \cancel{y}\right)}{(y-8)^3} =$$

$$= \underbrace{\frac{2y}{(y-8)^3}}_{>0}\left((y-8)^3 - 8\right) = 0 \Leftrightarrow (y-8)^3 = 8 \; ; \quad y - 8 = 2$$

$$\underline{\underline{y = 10}}$$

$$(x = 5, \quad l = \sqrt{125})$$

A kapott értéknél valóban minimum, hiszen

$$\lim_{y \to 8+} l(y) = \lim_{y \to \infty} l(y) = \infty, \quad \text{és } l \text{ folytonos.}$$

Pontozás:

Hasonlóság: 3 p.

l felírása egy paraméterrel: 4 p.

deriválás: 4 p.

derivált zérushelye: 3 p.

ez minimum: 2 p.

B VARIÁNS

Csak eredményt; a részletes pontozás az α variánsnál azonos.

1, i, $\boxed{17}$

$$\int \frac{1}{3-e^x}\,dx = \int \frac{1}{3-u}\cdot\frac{du}{u}\Big|_{u=e^x} \;(5)\; = \int \frac{1/3}{3-u} + \frac{1/3}{u}\,du = \underline{\underline{\frac{1}{3}\ln|3-e^x| + \frac{x}{3} + C}}$$

(3) (4) (3)+(2)

ii, $\boxed{8}$

$$\int \frac{(x+3)^2}{x^2+3^2}\,dx = \int\left(1+\frac{6x}{x^2+9}\right)dx \;(4)\; = \underline{\underline{x + 3\ln(x^2+9) + C}}$$

(1) (3)

— ∘ —

2, $\boxed{10}$

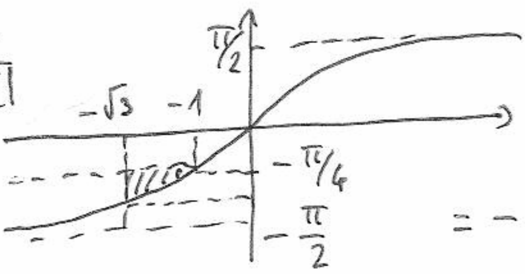
$$\int_4^\infty f(x)\,dx = \lim_{\Omega\to\infty}\int_4^\Omega f(x)\,dx \quad (3)$$

$$\int_4^\infty \frac{d}{dx}\left(x \sin\left(\frac{2\pi}{x}\right)\right)dx = \lim_{\Omega\to\infty} \Omega \sin\left(\frac{2\pi}{\Omega}\right) - 4\underbrace{\sin\left(\frac{\pi}{2}\right)}_{1} = \underline{\underline{2\pi - 4}}$$

(2) (4) (1)

— ∘ —

3, $\boxed{15}$



$$T = \int_{x=-\sqrt{3}}^{-1}\left(-\frac{\pi}{4} - \operatorname{arctg} x\right)dx \;(6)\; = \;(3)$$

$$= -\frac{\pi}{4}(\sqrt{3}-1) \;(1)\; - \overbrace{\Big[x \operatorname{arctg} x\Big]_{-\sqrt{3}}^{-1} + \int_{-\sqrt{3}}^{-1}\frac{x}{1+x^2}\,dx}^{} =$$

$$= \underbrace{-\frac{\pi}{4}(\sqrt{3}-\cancel{1}) - \cancel{\frac{\pi}{4}} + \frac{\sqrt{3}\pi}{3}}_{(2)} + \frac{1}{2}\underbrace{\Big[\ln(1+x^2)\Big]_{-\sqrt{3}}^{-1}}_{\ln 2 - \ln 4} \;(3)\; = \underline{\underline{\sqrt{3}\pi\cdot\frac{1}{12} - \ln\sqrt{2}}}$$

— • —

4, Lásd α,

— ∘ —

5, α-nál az egy tengelyre tükrözöttje. $D_f = (0,\infty)$ (1) ; $f(+\infty) = -\infty$ (2)

$f(0+) = 0$ (5)

$f'(x) = -\ln x\,(\ln x + 2)$ (3) $\quad f'(x) = 0 \iff x = 1 \vee x = e^{-2}$ (3)

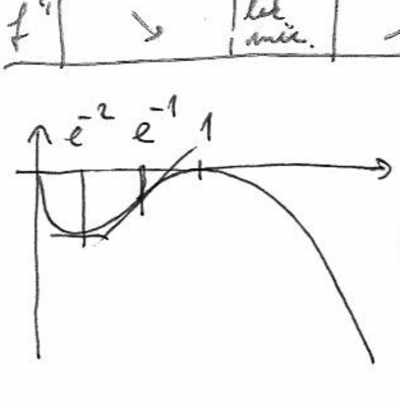
$f''(x) = \dfrac{-2(\ln x + 1)}{x}$ (2) ; $f''(x) = 0 \iff x = e^{-1}$ (2)

x	$(0, e^{-2})$	e^{-2}	$(e^{-2}, 1)$	1	$(1, \infty)$
f'	−	0	+	0	−
f	↘	lok. min.	↗	lok. max.	↘

(3)

x	$(0, e^{-1})$	e^{-1}	(e^{-1}, ∞)
f''	+	0	−
f	∪	infl. pont	∩

(2)



(5)

5 $\quad n \in \mathbb{N}^+$

$\text{arctg}(x+2) = 2\cos(\pi x)$ -n van megoldása $\left(2n - \frac{1}{2},\ 2n\right)$ -en?

2

Legyen $f(x) = \text{arctg}(x+2) - 2\cos(\pi x)$.

$$f\left(2n - \tfrac{1}{2}\right) = \text{arctg}\left(2n - \tfrac{1}{2} + 2\right) - 2\cos\left(2n\pi - \tfrac{\pi}{2}\right) = \text{arctg}\left(\overbrace{2n + \tfrac{3}{2}}^{>0}\right) > 0 \quad 2$$

$$f(2n) = \text{arctg}(2n+2) - 2\cos(2n\pi) = \underbrace{\text{arctg}(2n+2)}_{<\frac{\pi}{2}} - 2 < 0 \quad 2$$

$\Rightarrow f\left(2n - \frac{1}{2}\right) < 0 \quad f(2n) > 0$, így a Bolzano-tétel miatt a fv felveszi a 0-t a $\left(2n - \frac{1}{2},\ 2n\right)$ intervallumon, hiszen $f(x)$ folytonos. Ekkor $\exists\, \xi \in \left(2n - \frac{1}{2},\ 2n\right)$, melyre $f(\xi) = 0$, így

$\text{arctg}(\xi + 2) = 2\cos(\pi \xi)$, azaz van megoldás. 4