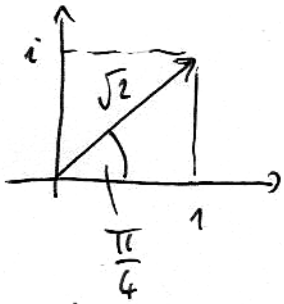
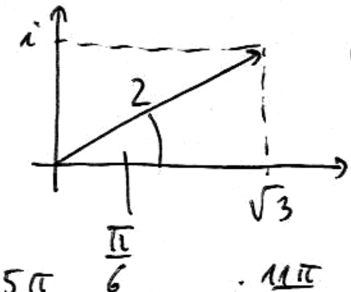
AN1 PZH1 MEGOLDÁS | α | -1- | 2017. XI. 6.

1, a, $\boxed{13}$ $\boxed{8}$



$1+i = \sqrt{2}\cdot e^{i\frac{\pi}{4}}$ (2)



$\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$ (3)

$$(1+i)^4(\sqrt{3}+i)^5 = (\sqrt{2})^4 e^{i\pi}\cdot 2^5\cdot e^{i\frac{5\pi}{6}} = 2^7\cdot e^{i\frac{11\pi}{6}} = \underline{\underline{128\cdot e^{-i\frac{\pi}{6}}}} \quad (3)$$

b, $\boxed{5}$

$$\frac{i}{1+\sqrt{3}i} = \frac{e^{i\frac{\pi}{2}}\ (1)}{2e^{i\pi/3}\ (2)} = \frac{1}{2}e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}\right)} = \underline{\underline{\frac{1}{2}e^{i\frac{\pi}{6}}}} \quad (2)$$

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2, $\bar{z} = z^n$; $z\in\mathbb{C}$; $n\in\mathbb{N}$

$\boxed{12}$ $\underline{n=0 \text{ esetén}}$: $z^n = 1$, tehát csak a $\underline{\underline{z=1}}$ a megoldás (1)

$\underline{n\geq 1 \text{ esetén}}$: $z = re^{i\varphi}$; $\bar{z} = re^{-i\varphi}$; $z^n = r^n e^{+in\varphi}$

$$\bar{z} = z^n \iff re^{-i\varphi} = r^n e^{in\varphi} \quad (3)$$

i, $r = r^n = 0 \Rightarrow r = 0 \Rightarrow \underline{\underline{z=0}}$ (2)

ii, $0 < r = r^n \Rightarrow r = 1$ (2)

és $2k\pi - \varphi = n\varphi \Rightarrow \varphi_k = \frac{2k\pi}{n+1}$ (3) $(k\in\{0,1,2,\dots n\}$

$$\underline{\underline{z_k = e^{i\frac{2k\pi}{n+1}}}} \quad (1)$$

Tehát $n=0$ esetén $z=1$ a megoldás, $n\geq 1$ esetén $z=0$ és $z_k = e^{i\frac{2k\pi}{n+1}}$, $k\in\{0..n\}$.

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3, a, $\boxed{13}$ $\boxed{5}$ $\lim_{n\to\infty} a_n = -\infty$, ha $\forall P\in\mathbb{R}$ (vagy $\forall P>0$) esetén $\exists N(P)\in\mathbb{N}$, melyre $\forall n>N(P)$ esetén $a_n \leq -P$.

b, $\boxed{8}$

$$|a_n - A| = |\sqrt{n+1}-\sqrt{n}| \ (2) = \frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}} \ (2) \leq \frac{1}{2\sqrt{n}} \ (2) < \varepsilon, \text{ ha}$$

$n > \left(\frac{1}{2\varepsilon}\right)^2$ (2); tehát az $N(\varepsilon) = \left[\frac{1}{4\varepsilon^2}\right]+1$ küszöbindex megfelel.

4, a,

$\boxed{8}$ $$a_n = \frac{1+2+\dots+n}{n+2} - \frac{n}{2} = \frac{n(n+1)}{2(n+2)} - \frac{n}{2} = \frac{n(n+1)-n(n+2)}{2(n+2)} =$$ (3)

$$= \frac{n^2+n-(n^2+2n)}{2(n+2)} = \frac{-n}{2n+4} \xrightarrow[n\to\infty]{} \underline{\underline{-\frac{1}{2}}}$$ (3) (2)

b,

$\boxed{7}$ $$b_n = \frac{2n^3}{2n^2+3} + \frac{-5n^2}{5n+1} = \frac{2n^3(5n+1)-5n^2(2n^2+3)}{(2n^2+3)(5n+1)} =$$ (3)

$$= \frac{\cancel{10n^4}+2n^3-\cancel{10n^4}-15n^2}{10n^3+2n^2+15n+3} \xrightarrow[n\to\infty]{} \frac{2}{10} = \underline{\underline{\frac{1}{5}}}$$ (2) (2)

c,

$\boxed{7}$ $$c_n = \left(\frac{2n+3}{n+1}\right)^n = 2^n\cdot\left(\frac{n+\frac{3}{2}}{n+1}\right)^n = 2^n\,\frac{\left(1+\frac{3/2}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n} \xrightarrow[n\to\infty]{} \infty$$ (3) (1)

$2^n \to \infty$ (1); $\left(1+\frac{3/2}{n}\right)^n \to e^{3/2}$ (1); $\left(1+\frac{1}{n}\right)^n \to e$ (1)

d,

$\boxed{8}$ $$\frac{1}{\sqrt[n]{3}}\cdot\frac{5}{2} = \sqrt[n]{\frac{5^n}{2\cdot 2^n+2^n}} < d_n = \sqrt[n]{\frac{5^n+n^2}{2n^2+2^n}} < \sqrt[n]{\frac{2\cdot 5^n}{2^n}} = \sqrt[n]{2}\,\frac{5}{2} \to \frac{5}{2}$$ (3) (3)

$\frac{1}{\sqrt[n]{3}}\cdot\frac{5}{2} \to \frac{5}{2}$; $\sqrt[n]{2} \to 1$

Rendőr-elv alapján $\lim\limits_{n\to\infty} d_n = \underline{\underline{\frac{5}{2}}}$. (2)

5, $a_1 = \sqrt{2}$; $a_{n+1} = \sqrt{a_n+2}$

$\boxed{6}$ a, Látható, hogy $\forall n$-re $a_n > 0$.

Teljes indukcióval igazoljuk, hogy $\forall n$-re $a_n < 2$.

i, $a_1 = \sqrt{2} < 2$ ✓ ii, T.f.h. $a_n < 2 \Rightarrow a_n + 2 < 4 \Rightarrow$

$a_{n+1} = \sqrt{a_n+2} < \sqrt{4} = 2$ ✓

$\boxed{6}$ b, Teljes indukcióval:

i, $a_1 = \sqrt{2} < a_2 = \sqrt{\sqrt{2}+2}$

ii, T.f.h. $a_n < a_{n+1} \Rightarrow 0 < a_n + 2 < a_{n+1} + 2 \Rightarrow \sqrt{a_n+2} = a_{n+1} < \sqrt{a_{n+1}+2} =$
$= a_{n+2}$ ✓

$\boxed{6}$ c, a_n korlátos, és monoton, tehát létezik $A = \lim\limits_{n\to\infty} a_n$ határérték (2)

(4) $\begin{cases} \text{Ekkor } 0 < A = \sqrt{A+2} \Rightarrow A^2 - A - 2 = (A-2)(A+1) = 0 \Rightarrow A_1 = 2 \text{ ✓},\ A_2 = -1 \text{ ↯} \\ \text{Tehát a határérték: } \underline{A = 2} \end{cases}$

6. $\boxed{14}$ $\cos(\pi n) = (-1)^n = \begin{cases} 1, & \text{ha } n \text{ páros} \\ -1, & \text{ha } n \text{ páratlan} \end{cases}$ (4)

Ha n páros: $a_n = \dfrac{4^{n-1}}{9^n + 9^n + 4^n} = \dfrac{1/4}{2\left(\frac{9}{4}\right)^n + 1} \longrightarrow 0$ (4)

$\left(\frac{9}{4}\right)^n \to \infty$

Ha n páratlan: $a_n = \dfrac{4^{n-1}}{\cancel{-9^n} + \cancel{9^n} + 4^n} = \dfrac{1}{4}$ (3)

Tehát $\liminf a_n = 0$ (1), $\limsup a_n = \frac{1}{4}$ (1), és nem létezik a limesz. (1)

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IMSc $\boxed{8}$ $r > 0$; $a_n \to +0$; $b_n \to 0$

$a_n^{b_n} = e^{b_n \cdot \ln a_n} \longrightarrow r = e^{\ln r} \iff \underset{\downarrow\, 0}{b_n} \cdot \underbrace{\ln \overset{\uparrow\, +0}{a_n}}_{\downarrow\, (-\infty)} \to \ln r$ (4)

Legyen pl. $a_n = \frac{1}{n}$, $b_n = \dfrac{\ln r}{\ln(1/n)} = \dfrac{-\ln r}{\ln n} = -\log_n r$ $(n \geq 2)$

Ekkor valóban $a_n^{b_n} = \left(\frac{1}{n}\right)^{-\log_n r} = n^{\log_n r} = r$. (4)

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Vagy $a_n = e^{-n}$, $b_n = \dfrac{\ln r}{\ln a_n} = \dfrac{\ln r}{-n} = -\dfrac{\ln r}{n}$

Ekkor is $a_n^{b_n} = \left(e^{-n}\right)^{-\frac{\ln r}{n}} = r$ vagy (4)