Űrkommunikáció
Space Communication
2023/11.

Optimal receiver (demodulator)

- MAP – Maximum A-posteriori Probability receiver (absolute optimal receiver)

$$\hat{s}_i(t) = arg \max_{s_i(t)\epsilon S}\{f^D(s_i(t) \mid r(t))\};\ with\ a-posteriori\ f^D(s_i(t) \mid r(t))$$

Or equivalently signal vector space

$$\hat{\bar{s}}_i = arg \max_{\bar{s}_i\,\epsilon\,\bar{S}}\{f(\bar{s}_i \mid \bar{r})\};$$

$$f(\hat{\bar{s}}_i \mid \bar{r}) \geq f(\bar{s}_i \mid \bar{r}) \quad \forall\ \bar{s}_i\ \epsilon\ \bar{S}$$

Equivalently whit a-priori probability $p(\bar{s}_i) = p(s_i(t)) = p(m_i)$

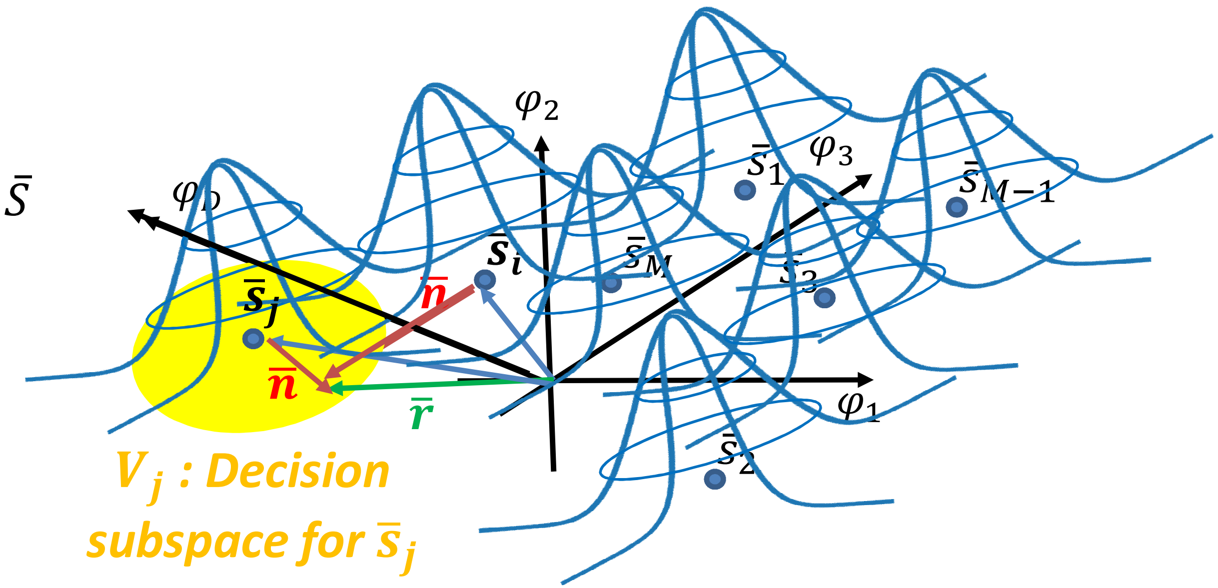
$$\hat{\bar{s}}_i = arg \max_{\bar{s}_i\,\epsilon\,\bar{S}}\{f(\bar{r} \mid \bar{s}_i) \cdot p(\bar{s}_i)\};$$

$$f(\bar{r} \mid \hat{\bar{s}}_i) \cdot p(\hat{\bar{s}}_i) \geq f(\bar{r} \mid \bar{s}_i) \cdot p(\bar{s}_i) \quad \forall\ \bar{s}_i\ \epsilon\ \bar{S}$$

- ML – Maximum Likelihood receiver (suboptimal receiver). If $p(m_i)$ uniform, then MAP ≡ ML

$$\hat{\bar{s}}_i = arg \max_{\bar{s}_i\,\epsilon\,\bar{S}}\{f(\bar{r} \mid \bar{s}_i)\}$$

$$f(\bar{r} \mid \hat{\bar{s}}_i) \geq f(\bar{r} \mid \bar{s}_i) \quad \forall\ \bar{s}_i\ \epsilon\ \bar{S}$$



Optimal vectorial receiver

MAP: $f^D(\bar{r}\mid\hat{\bar{s}}_i)\cdot p(\hat{\bar{s}}_i) \geq f^D(\bar{r}\mid\bar{s}_i)\cdot p(\bar{s}_i) \quad \forall\ \bar{s}_i \in \bar{S}$

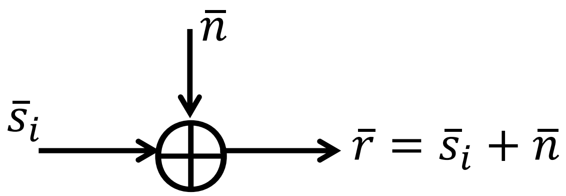
P_{c_i}*: Probability of correct decision for* $\bar{s}_i$ *with decision subspace* V_i

$$P_{c_i} = \oiiint_{V_i} f^D(\bar{r}\mid\hat{\bar{s}}_i)\,d\bar{r}$$

P_e*: Error Probability of MAP receiver*

$$P_e = 1 - \sum_{i=1}^{M} p(\bar{s}_i)\cdot P_{c_i}$$

Optimal vectorial receiver over AWGN



D=1 and D dimensional distribution of the noise

$$f_n^1(n) = \frac{1}{\sqrt{2\pi\sigma_n^2}} exp\left(-\frac{n^2}{2\sigma_n^2}\right); G\left(\mu_n = 0, \sigma_n^2 = {}^{N_0}/_{2}\right); \quad f_{\bar{n}}^D(n) = \frac{1}{(\pi\cdot N_0)^{D/2}} exp\left(-\frac{|\bar{n}|^2}{N_0}\right)$$

D dimensional conditional distribution of the received vector

$$f^D(\bar{r}\mid\hat{\bar{s}}_i) = \frac{1}{(\pi\cdot N_0)^{D/2}} exp\left(-\frac{\left|\overbrace{\bar{r}-\hat{\bar{s}}_i}^{\bar{n}}\right|^2}{N_0}\right)$$

Optimal vectorial receiver over AWGN

$$\textit{MAP: } f^D(\bar{r} \mid \hat{\bar{s}}_i) \cdot p(\hat{\bar{s}}_i) \geq f^D(\bar{r} \mid \bar{s}_i) \cdot p(\bar{s}_i) \quad \forall \ \bar{s}_i \in \bar{S}$$

Maximum of

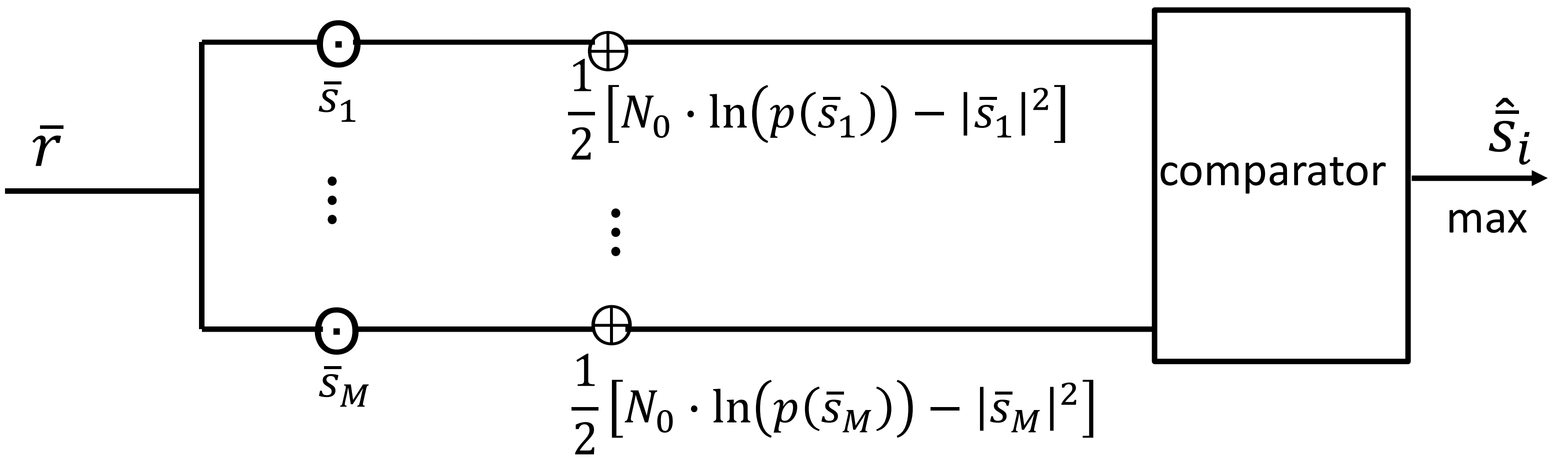
$$f^D(\bar{r} \mid \bar{s}_i) \cdot p(\bar{s}_i) = p(\bar{s}_i) \cdot \frac{1}{(\pi \cdot N_0)^{D/2}} exp\left(- \frac{\left|\overbrace{\bar{r} - \bar{s}_i}^{\bar{n}}\right|^2}{N_0} \right) \overset{\ln(.);\ \cdot N_0}{\Longleftarrow}$$

$$= N_0 \cdot \ln(p(\bar{s}_i)) - \underbrace{\frac{D}{2} \cdot N_0 \cdot \ln(\pi \cdot N_0)}_{Constant} - |\bar{r} - \bar{s}_i|^2 \overset{(\cdot)^2}{\Longrightarrow}$$

$$= N_0 \cdot \ln(p(\bar{s}_i)) - |\bar{r}|^2 + 2 \cdot \bar{r} \cdot \bar{s}_i - |\bar{s}_i|^2 \xrightarrow{\frac{1}{2};\ |\bar{r}|^2 indifferent\ by\ \mathrm{ma}\ \bar{\ }_i}$$

$$= \bar{r} \cdot \bar{s}_i + \frac{1}{2}\left[N_0 \cdot \ln(p(\bar{s}_i)) - |\bar{s}_i|^2\right]$$

Optimal vectorial receiver

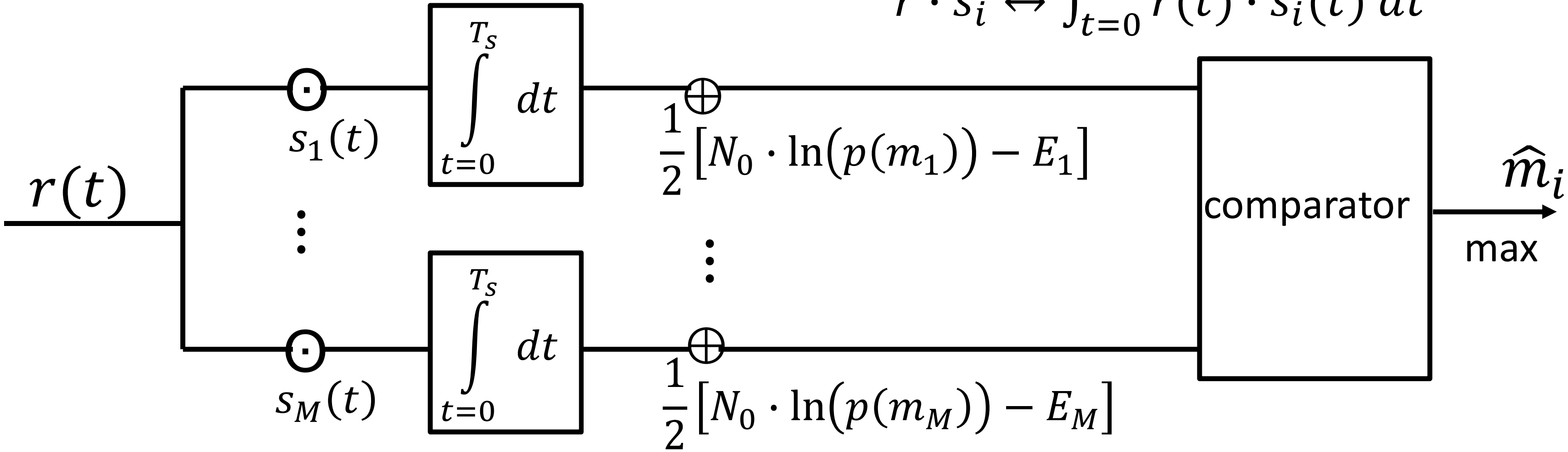


Optimal receiver over AWGN

Correlation receiver

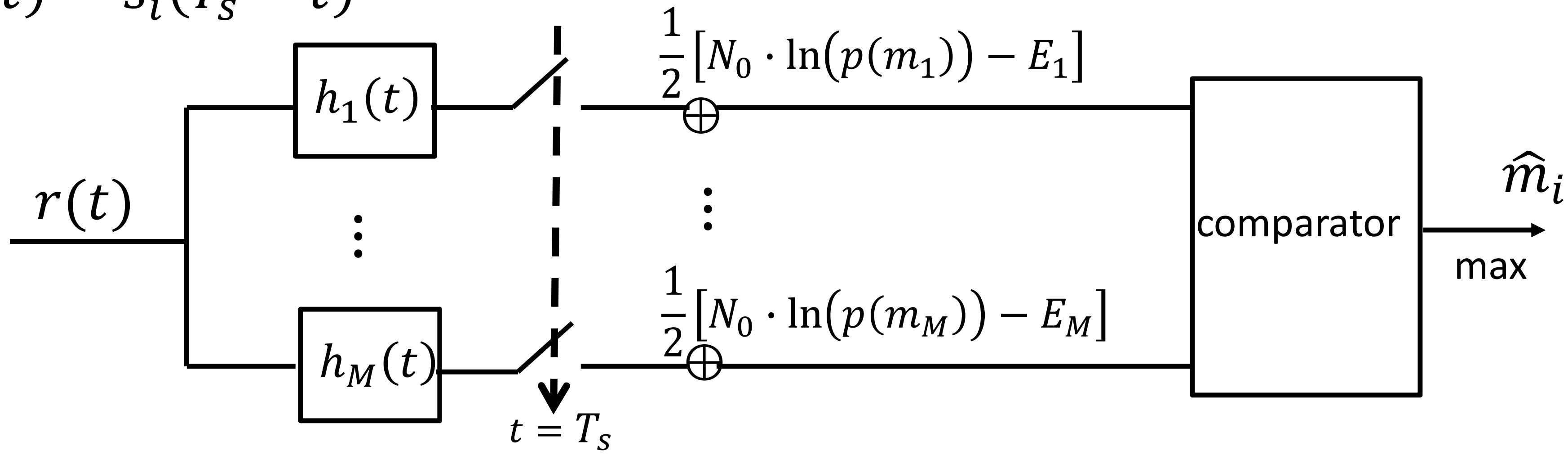
scalar product = correlation

$$\bar{r} \cdot \bar{s}_i \leftrightarrow \int_{t=0}^{T_s} r(t) \cdot s_i(t)\, dt$$



Matched Filter receiver: $\int_{t=0}^{T_s} r(t) \cdot s_i(t)\, dt = \int_{-\infty}^{\infty} r(\tau) \cdot s(T_s - t + \tau) d\tau \quad t = T_s$

$$h_i(t) = s_i(T_s - t)$$



„regular simplex" signal constellation

M-ary digital modulation with energy constrain E_{max}

- M different signal waveform
- $E_i \leq E_{max}$.

Optimal signal set ↔„regular simplex" signal vector constellation

$S = \{s_1(t), \dots, s_M(t)\} \leftrightarrow \bar{S} = \{\bar{s}_1, \dots, \bar{s}_M\}$.

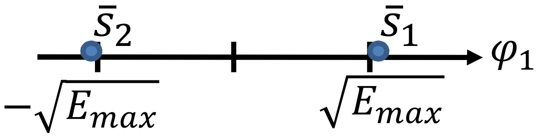
Def.: of „regular simplex" vector constellation

Dimension: D=M-1; Energy: $|\bar{s}_i|^2 = E_{max}$ $\forall\, i$; Correlation: $\bar{s}_i \cdot \bar{s}_j = -\frac{E_{max}}{M-1}$ $\forall\, i, j \quad i \neq j$

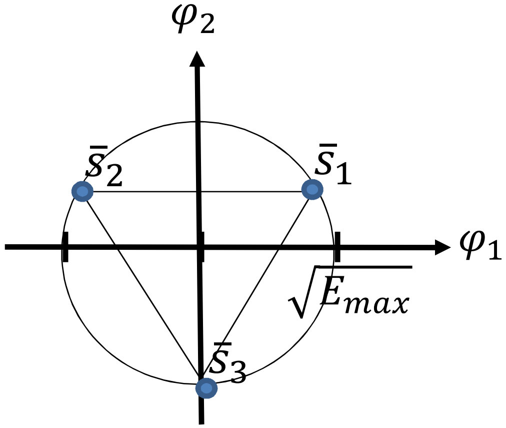
Examples:

M=2 => D=1

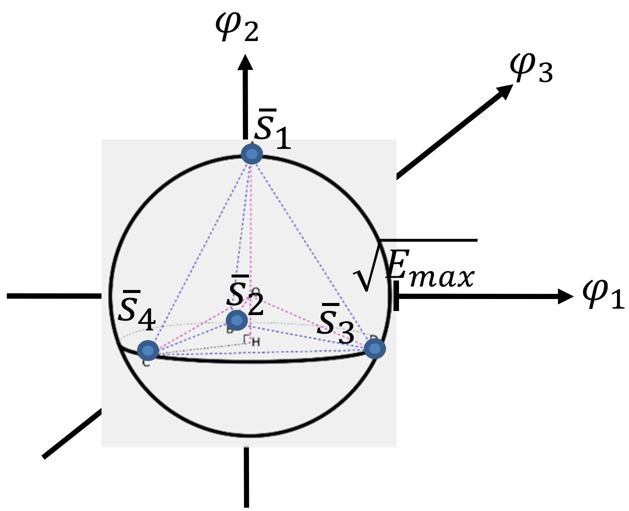
Antipodal signals



M=3 => D=2



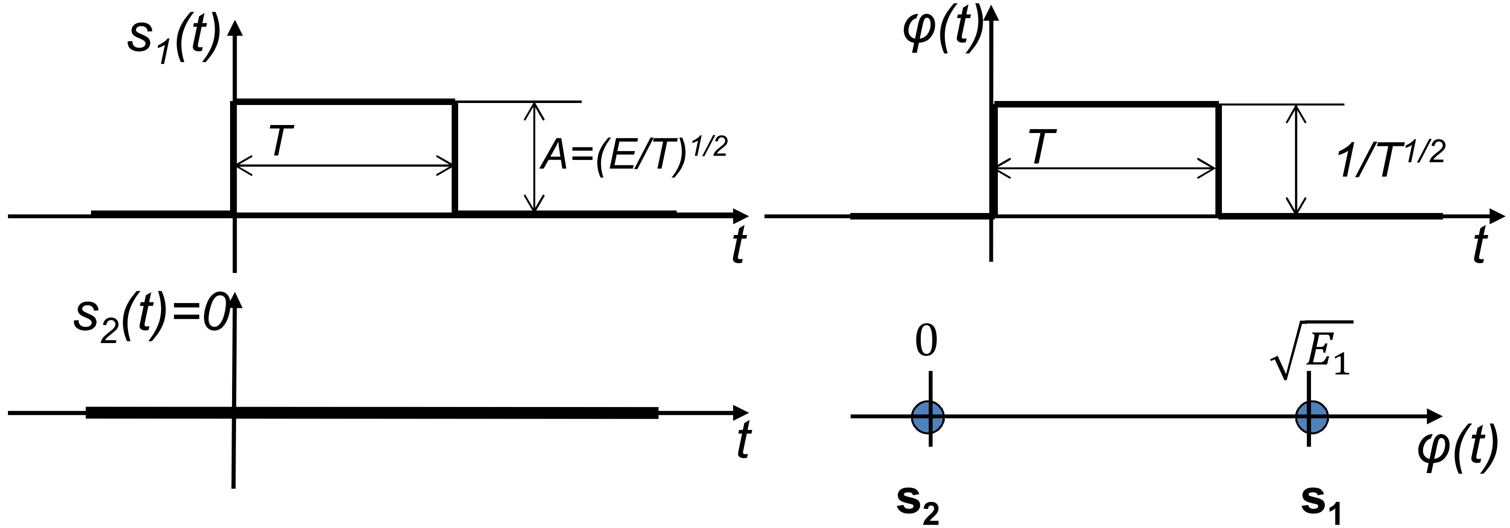
M=4 => D=3

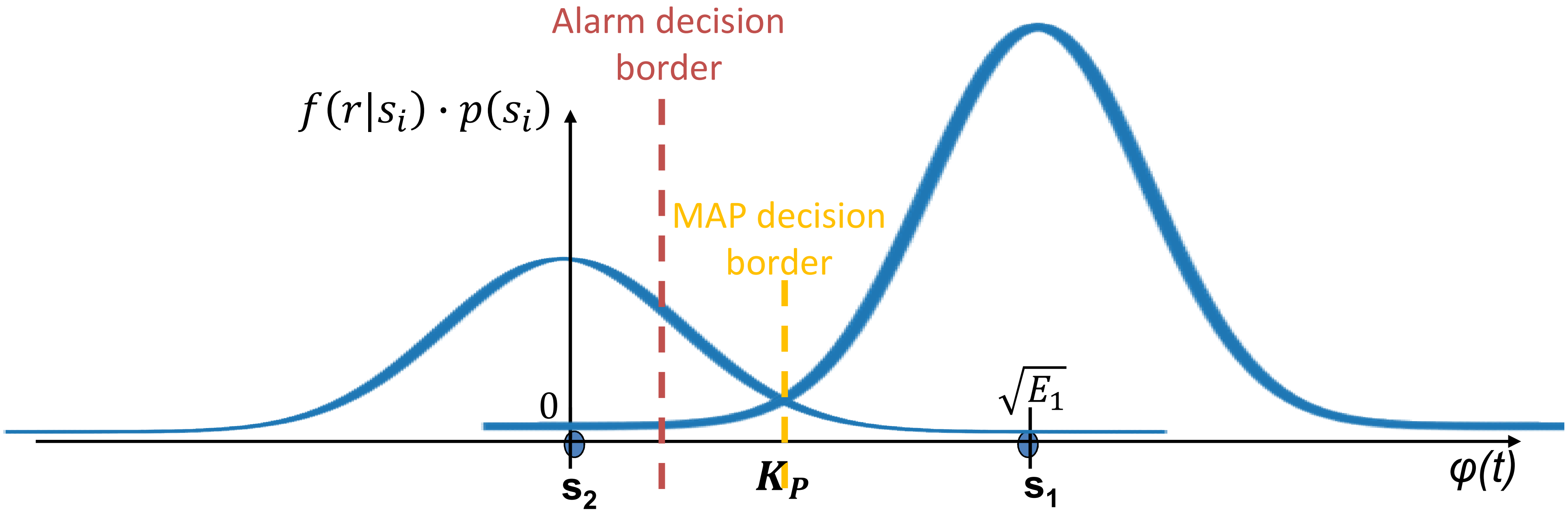


etc....

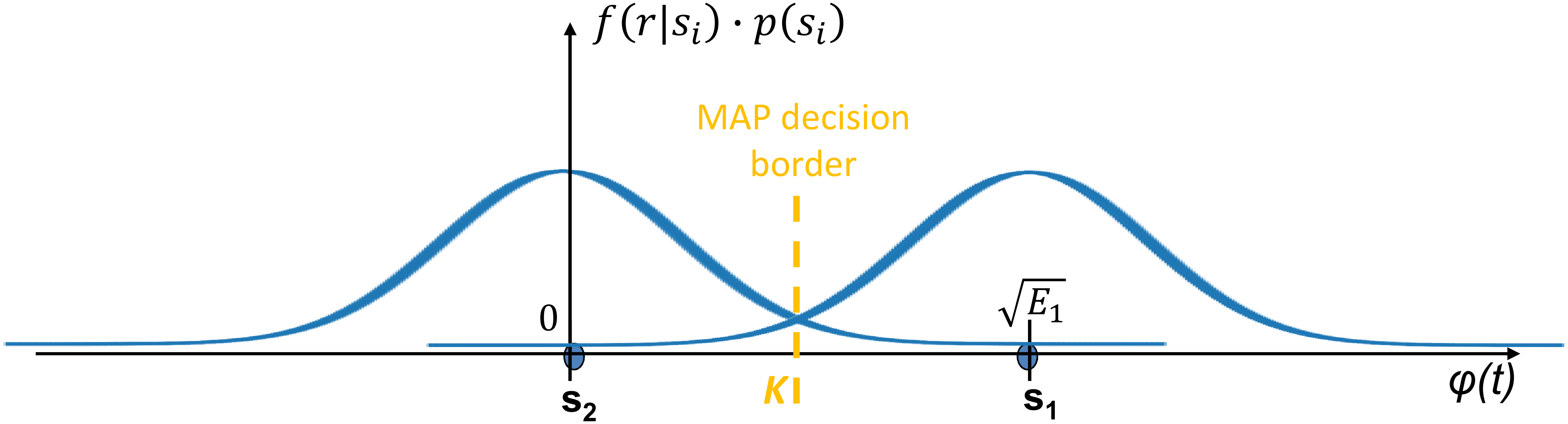
Baseband digital modulation, OOK

On-off keying (OOK), M=2, D=1, $\mathrm{P} = \{p(m_1) > p(m_2)\}$





On-off keying (OOK), M=2, D=1, $P = \{p(m_1) = p(m_2) = 1/2\}$



Define border *K* of MAP decision:

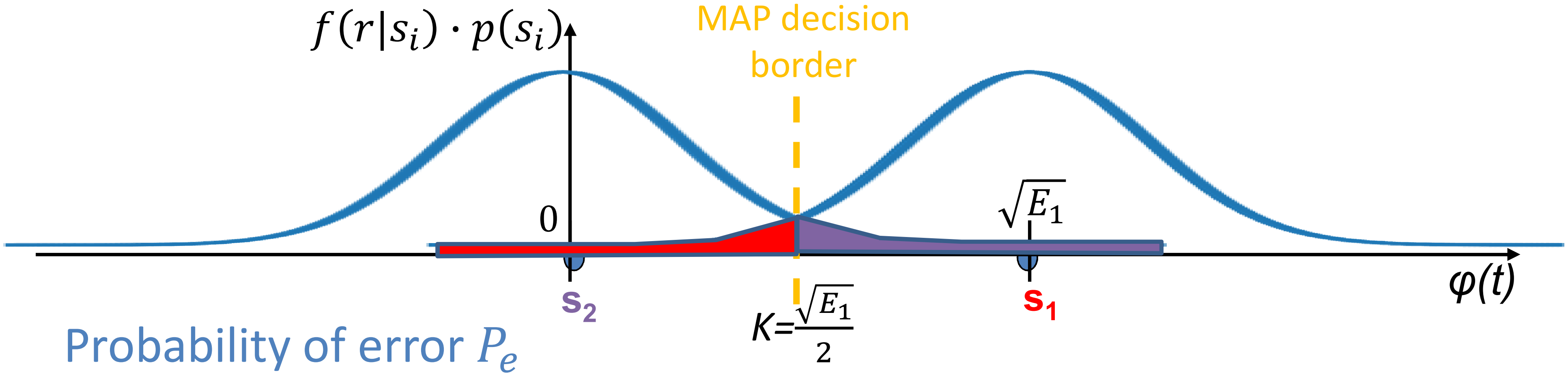
$$\frac{1}{2} \cdot \frac{1}{(\pi \cdot N_0)^{1/2}} exp\left(-\frac{K^2}{N_0}\right) = \frac{1}{2} \cdot \frac{1}{(\pi \cdot N_0)^{1/2}} exp\left(-\frac{\left(K - \sqrt{E_1}\right)^2}{N_0}\right)$$

$$-\frac{K^2}{N_0} = -\frac{\left(K - \sqrt{E_1}\right)^2}{N_0}$$

$$K^2 = K^2 - 2 \cdot K \cdot \sqrt{E_1} + E_1$$

$$K = \frac{\sqrt{E_1}}{2}$$

On-off keying (OOK), M=2, D=1, P $= \{p(m_1) = p(m_2) = 1/2\}$



Probability of error P_e

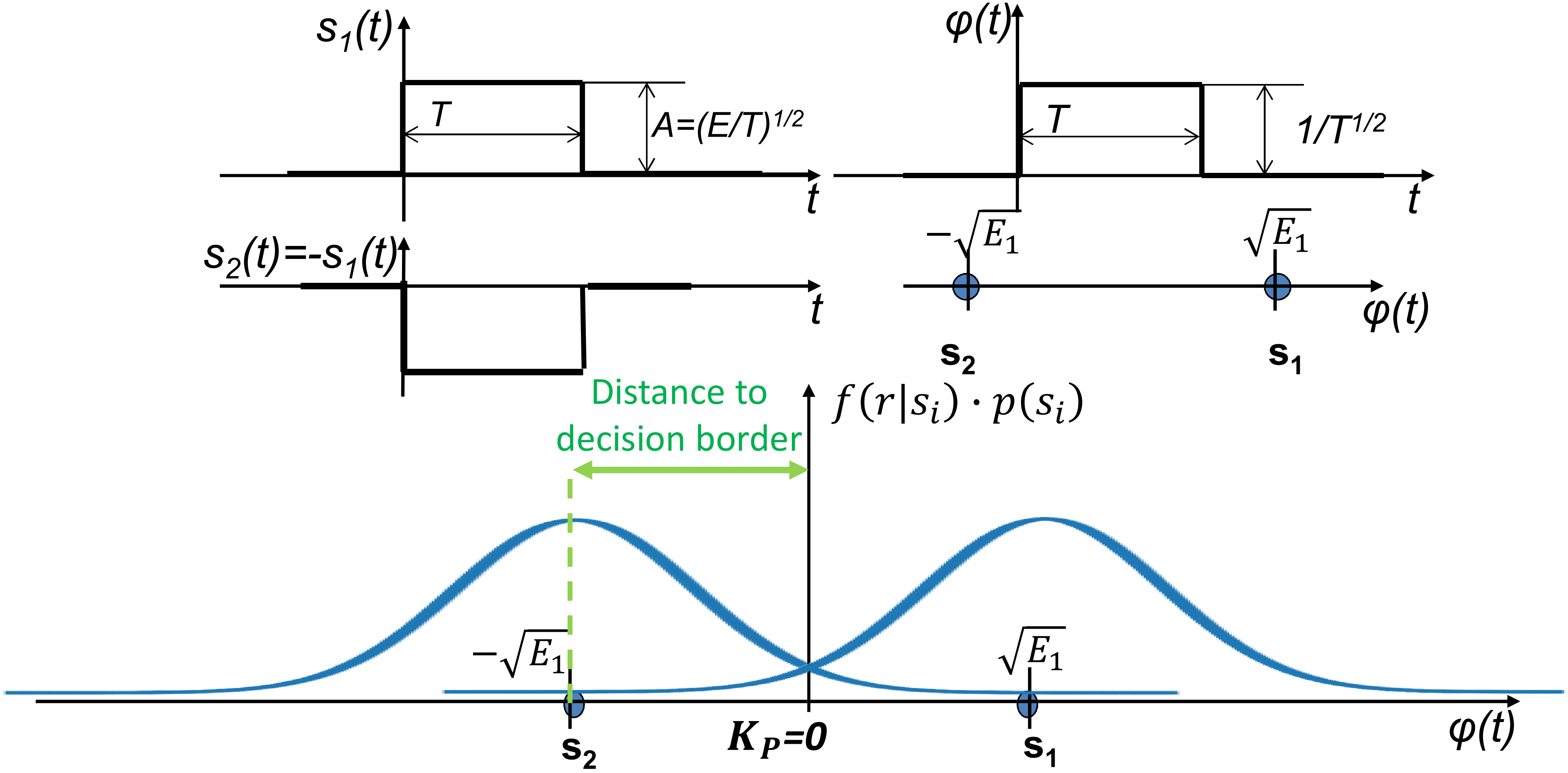
$$P_e = 2 \cdot \int_{\frac{\sqrt{E_1}}{2}}^{\infty} \frac{1}{2} \cdot \frac{1}{(\pi \cdot N_0)^{1/2}} exp\left(-\frac{n^2}{N_0}\right) dn = \frac{1}{\sqrt{\pi}} \cdot \int_{\frac{\sqrt{E_1}}{2}}^{\infty} \frac{1}{\sqrt{N_0}} exp\left(-\left(\frac{n}{\sqrt{N_0}}\right)^2\right) dn$$

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \cdot \int_x^{\infty} exp(-u^2)\, du; subtitute\ u = \frac{n}{\sqrt{N_0}}, \dot{u} = \frac{1}{\sqrt{N_0}}$$

$$P_{e,OOK} = \frac{1}{\sqrt{\pi}} \cdot \int_{\frac{\sqrt{E_1/N_0}}{2}}^{\infty} exp\left(-\left(\frac{n}{\sqrt{N_0}}\right)^2\right) d\frac{n}{\sqrt{N_0}} = \frac{1}{2} \cdot erfc\left(\frac{1}{2} \cdot \sqrt{\frac{E_1}{N_0}}\right)$$

Baseband digital modulation, NRZ

Non-return-to-zero (NRZ), M=2, D=1, P $= \{p(m_1) = p(m_2)\}$



$$P_{e,NRZ} = \frac{1}{2}\cdot erfc\left(\sqrt{\frac{E_1}{N_0}}\right) \xLeftarrow{3dB\ imropvement} P_{e,OOK} = \frac{1}{2}\cdot erfc\left(\sqrt{\frac{E_{av}}{2N_0}}\right);$$

Bandpass Modulation or Passband Modulation

- Refers to modulation of signal over a carrier frequency
- According to the information symbol modulate the parameters of a harmonic function (e.g. sin or cos) at carrier frequency

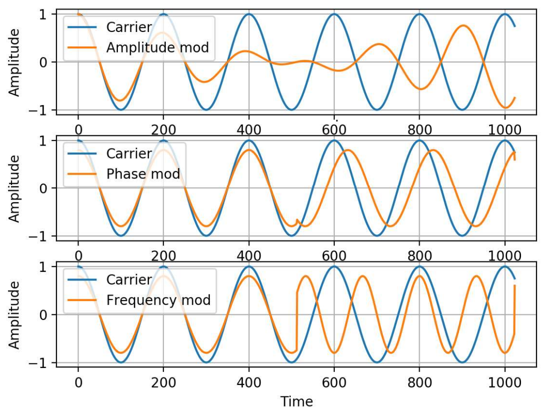
Unmodulated carrier

$$v(t) = \sqrt{2} \cdot A \cdot \cos[\omega_c t + \Phi]$$

- ***A***: Amplitude effective value
- $\boldsymbol{\omega_c} = 2\pi \cdot f_c$ Circular (angular, radial) frequency
- ***Φ*** : starting angle

General form of modulated carrier

$$x(t) = \sqrt{2} A d(t) \cos[\omega_c t + \vartheta(t) + \Phi]$$



Basic types of Modulation:

- *d(t)*: Amplitude modulation (Amplitude Shift Keying, ASK)
- *ϑ(t)*: Angular modulation (two type of)
  - Phase modulation: *ϑ(t) corresponds to the baseband information* (PSK)
  - Frequency modulation: *dϑ(t)/dt corresponds to the baseband information* (FSK)

Or combination of:

- amplitude and phase: e.g. QAM, APSK, etc.
- amplitude and frequency: biorthogonal modulation, etc.

Quadrature form, Complex Envelope

General form of modulated carrier

$$x(t) = \sqrt{2} A d(t) \cos[\omega_c t + \vartheta(t) + \Phi]$$

Suppose starting angle Φ=0 and applying

$$\sin x \cdot \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)];\ \cos x \cdot \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$x(t) = \sqrt{2} \cdot A \cdot d(t) \cdot [\cos(\vartheta(t)) \cdot \cos(\omega_c t) - \sin(\vartheta(t)) \cdot \sin(\omega_c t)]$$

Quadrature form:

$$x(t) = A \cdot a(t) \cdot \cos(\omega_c t) - A \cdot q(t) \cdot \sin(\omega_c t)$$

Where

$$a(t) = \sqrt{2} d(t) \cos \vartheta(t); q(t) = \sqrt{2} d(t) \sin \vartheta(t) \quad d(t) = \frac{\sqrt{a[(t)]^2 + q[(t)]^2}}{\sqrt{2}}; \vartheta(t) = arctg \frac{q(t)}{a(t)}$$

Complex Envelope:

$$x(t) = A \cdot \mathrm{Re}\{ \underbrace{[a(t) + j \cdot q(t)]}_{\text{Complex Envelope}} \cdot e^{j\omega_c t}\}$$

I, Inphase *and*

Q, Quadrature phase

Quality, Bandwidth and Price of Bandpass Digital Modulation

Quality: mostly two measure

- Probability of errors P_e
  - Bit Error Probability, Bit Error Rate – BER,
  - Symbol Error Rate – SER, Packet Error Rate – PER, etc.
- Speed of Information transfer
  - Bit/Symbol/Packet Rate – R [bit/sec, …]
- But also processing time by some real time apps

Bandwidth B of digital communication:

- Because of finite signal duration T_s theoretically infinite bandwidth (Recap: Fourier transf.)
- In practice:
  - At one carrier system – D=2 dimensional modulation (e.g. ASK, PSK, QAM):
    $B \approx 1/T_s$ because using filters (Recap: Matched Filter receiver)
  - At multiple carrier system – D=M dimensional modulation (e.g. FSK, OFDM):
    $B \approx M/T_s$

Price of digital communication:

- The necessary power: P
- The necessary bandwidth: B
- The necessary signal processing: Digital computation (negligible)
  - These can be converted by guarantying the same P_e: $B \downarrow \;\Rightarrow\; P \uparrow$ also $P \downarrow \;\Rightarrow\; B \uparrow$

  Using higher order modulation: instead of binary M=2^1, M=2^n-ary modulation

$$P_e \leftrightarrow SNR, \text{e.g.}\; E/N_0: \; E_1 = P \cdot T_b \leftrightarrow B \approx 1/T_b \;\text{vs.}\; E_n = P \cdot n \cdot T_b \leftrightarrow B \approx 1/nT_b$$

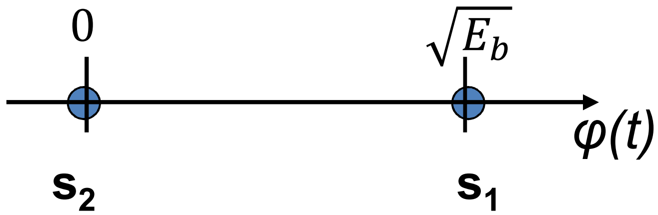
Binary Bandpass Modulation

- Binary Amplitude Shift Keying – BASK; M=2, D=1

$$s_1(t) = \sqrt{2} \cdot \sqrt{E_b / T_b} \cos \omega_c t$$

$$s_2(t) = 0$$

$$\varphi(t) = \sqrt{\frac{2}{T_b}} \cos \omega_c t$$



Bit error probability will be the same as by baseband OOK

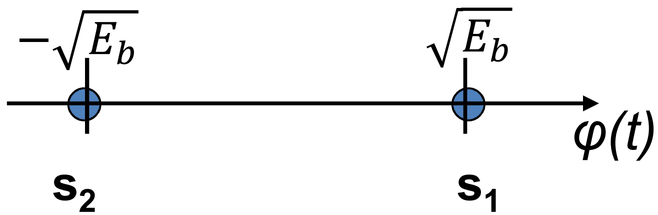
$$P_{e,BASK} = \frac{1}{2} \cdot erfc\left(\frac{1}{2} \cdot \sqrt{\frac{E_b}{N_0}}\right) = \frac{1}{2} \cdot erfc\left(\sqrt{\frac{E_{av}}{2N_0}}\right)$$

- Binary Phase Shift Keying – BPSK; M=2, D=1

$$s_1(t) = \sqrt{2} \cdot \sqrt{E_b / T_b} \cos \omega_c t$$

$$s_2(t) = \sqrt{2} \cdot \sqrt{E_b / T_b} \cos(\omega_c t + \pi)$$

$$\varphi(t) = \sqrt{\frac{2}{T_b}} \cos \omega_c t$$



Antipodal signal set – „regular simplex” for D=1

Bit error probability will be the same as by baseband NRZ

$$P_{e,BPSK} = \frac{1}{2} \cdot erfc\left(\sqrt{\frac{E_1}{N_0}}\right)$$

M-ary Phase Modulation

Set of Digital Signals: $S=\{s_1(t),\dots,s_M(t)\},\ \ M=2^n$

Symbol energy $E_s=E_i=\int_{t=0}^{T_s}s_i^2(t)dt \quad \forall\, i$

Symbol duration $T_s=n\cdot T_b$

Signal waveforms $s_i(t)\ \ for\ \ i=0,1,\dots,M-1$

$$s_i(t)=\sqrt{\frac{2\cdot E_s}{T_s}}\cos\left(\omega_c\cdot t+i\cdot\frac{2\pi}{M}+\varphi\right)$$

Basis functions of a D=2 dimensional signal space Φ

$$\varphi_1(t)=\sqrt{\frac{2}{T_s}}\cos(\omega_c\cdot t)\ \ and\ \ \varphi_2(t)=\sqrt{\frac{2}{T_s}}\sin(\omega_c\cdot t)$$

$$\int_{t=0}^{T_s}\varphi_1(t)\cdot\varphi_2(t)\,dt=\frac{2}{T_s}\int_{t=0}^{T_s}\cos(\omega_c\cdot t)\cdot\sin(\omega_c\cdot t)\,dt=\frac{1}{T_s}\int_{t=0}^{T_s}[\sin(2\omega_c\cdot t)-\sin(0)]dt$$

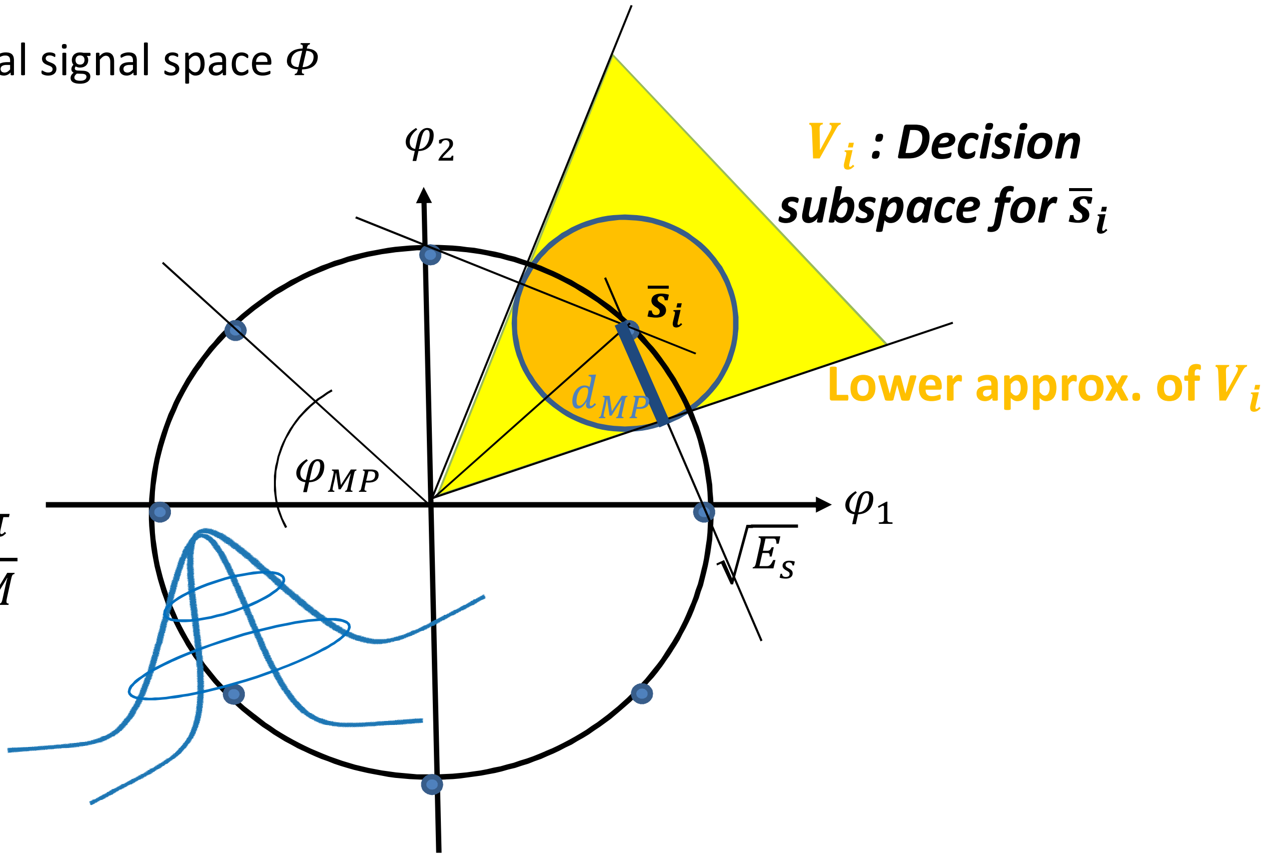
$$=\frac{1-\cos(2\omega_c T_s)}{2\omega_c T_s}=\begin{cases}0 & if\ \ 2\omega_c T_s=2k\pi\leftrightarrow\omega_c=\dfrac{k\pi}{T_s}\\[2ex] \cong 0 & if\ \ \omega_c\gg\dfrac{1}{T_s};quasi-orthogonal\end{cases}$$

Signal waveforms and vectors: $s_i(t)\leftrightarrow\bar{s}_i\ \ for\ \ i=0,1,\dots,M-1$

$$s_i(t)=s_{i1}\cdot\varphi_1(t)+s_{i2}\cdot\varphi_2(t)\leftrightarrow\bar{s}_i=[s_{i1},s_{i2}]$$

M-ary Phase Modulation

MPSK, D=2 dimensional signal space Φ



$$\varphi_{MP} = \frac{2\pi}{M}$$

$$d_{MP} = \sqrt{E_s}\sin\frac{\pi}{M}$$

Upper bound of symbol error probability $P_{e,s}$ (or SER)

$$P_{e,s} < 1 - \sum_{i=1}^{M} p(\bar{s}_i) \cdot \int_{\rho=0}^{d_{MP}} \int_{\varphi=0}^{2\pi} f(\bar{r} \mid \bar{s}_i) d\rho d\varphi$$