HÁZI CSOBA

1. Villamosmérnök szigorlat

MATEMATIKA A2
VILLAMOSMÉRNÖK HALLGATÓKNAK

2016. június 2.
Munkaidő: 90 perc

A szigorlaton semmilyen segédeszköz nem használható!

1. **(20 pont)**
Végezzen teljes függvényvizsgálatot az

$$f(x)=e^{\frac{1}{1-x}}$$

függvényen és vázolja fel a függvényt!

2. **(8+8+8+8 pont)**
Határozza meg a következő integrálokat!

(a)

$$\int_0^\infty \frac{e^x}{e^{2x}+1}\,\mathrm{d}x.$$

(b)

$$\int x\arctan x\,\mathrm{d}x.$$

(c)

$$\int_0^1\int_x^1 \frac{x\sin y}{y}\,\mathrm{d}y\,\mathrm{d}x.$$

(d)

$$\iint_T xy\,\mathrm{d}x\,\mathrm{d}y,$$

ahol T az $(x-1)^2+y^2=1$ és az $(x-2)^2+y^2=4$ körök által határolt korlátos tartomány $y=0$ és $y=x$ közötti része.

3. **(20 pont)**
Elemezze, hogy **a** és **b** paraméterek értékei mellett, mikor hány megoldása van az alábbi egyenletrendszernek!

$$\begin{aligned} x_1 - 3x_2 \qquad - 6x_4 &= 9 \\ 2x_1 + x_2 - 5x_3 + x_4 &= 8 \\ 2x_2 - x_3 + 2x_4 &= -5 \\ x_1 + 4x_2 - 7x_3 + \mathbf{a}x_4 &= \mathbf{b}. \end{aligned}$$

4. **(5+8 pont)**

(a) Határozza meg azokat a komplex számokat, melyeknek négyzete megegyezik a konjugáltjukkal!

(b) Határozza meg a

$$\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{2016} + \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{2016}$$

komplex szám algebrai alakját!

5. **(15 pont)**
Határozza meg az alábbi mátrix sajátértékeit, sajátvektorait és a determinánsát

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$

1/

1\. zárthelyi megoldásai (2016. június 3.)

(1)

$f(x) = e^{\frac{1}{1-x}}$

1, $D_f = \mathbb{R} \setminus \{1\}$, $f(x) > 0 \rightsquigarrow R_f = (0, \infty)$ $\boxed{1p}$

2) tengelymetszetek: ∘ $f(x) \neq 0$ $\boxed{1p}$

∘ $f(0) = e$

3) viselkedés az ÉT szélein:

∘ $\lim_{x \to -\infty} e^{\overbrace{\frac{1}{1-x}}^{\to 0}} = 1$ $\boxed{2p}$

∘ $\lim_{x \to \infty} e^{\overbrace{\frac{1}{1-x}}^{\to 0}} = 1$ $\boxed{2p}$

∘ $\lim_{x \to 1-0} e^{\frac{1}{1-x}} = \infty$ $\boxed{2p}$

$\uparrow$

$\xrightarrow{} \underset{1}{+}$ $\quad \frac{1}{1-x} = +\infty$, $1-x \to 0$, >0

∘ $\lim_{x \to 1+0} e^{\frac{1}{1-x}} = 0$ $\boxed{2p}$

$\underset{1}{+}\xleftarrow{}$ $\quad \frac{1}{1-x} = -\infty$, $1-x \to 0$, <0

(2) 4) monotonitás

$f'(x) = e^{\frac{1}{1-x}} \cdot \left(-(1-x)^{-2}\right) \cdot (-1) = \frac{1}{(1-x)^2} e^{\frac{1}{1-x}} > 0$, ha $x \neq 1$

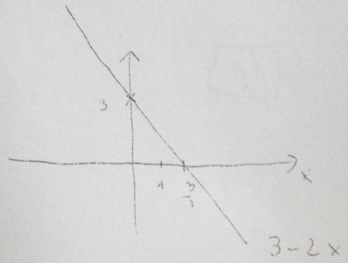
$\Rightarrow \boxed{f(x) \nearrow}$ $\boxed{3p}$

2/

5) konvexita

$$f''(x) = 2\,\frac{1}{(1-x)^3}\,e^{\frac{1}{1-x}} + \frac{1}{(1-x)^2}\cdot\frac{1}{(1-x)^2}\,e^{\frac{1}{1-x}} =$$

$$= \frac{1}{(1-x)^3}\,e^{\frac{1}{1-x}}\underbrace{\left[2+\frac{1}{1-x}\right]}_{\frac{2-2x+1}{(1-x)}} = \underbrace{\frac{1}{(1-x)^4}\,e^{\frac{1}{1-x}}}_{>0}(3-2x)$$

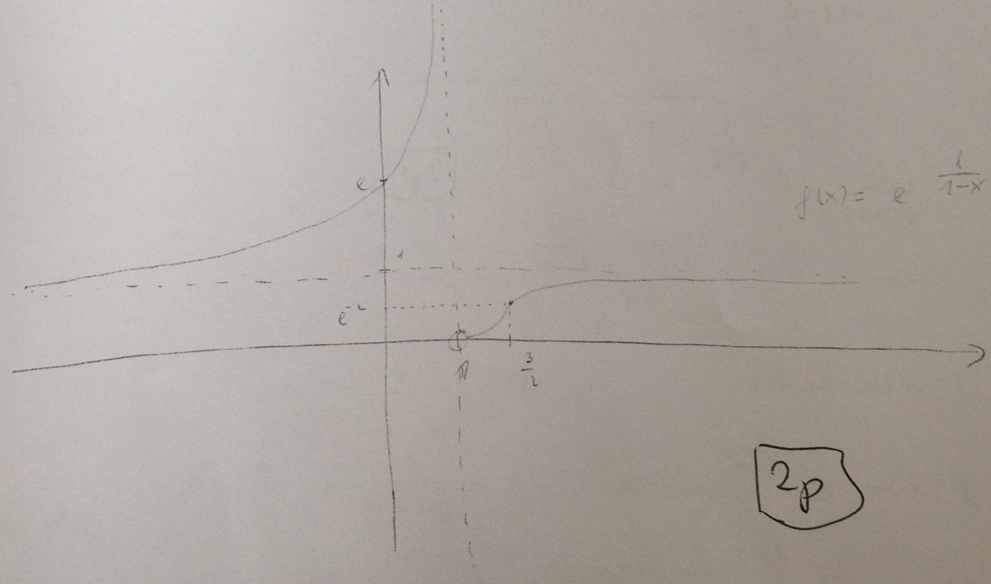


$\boxed{3p}$

x	$(-\infty, 1)$	1	$(1, \frac{3}{2})$	$\frac{3}{2}$	$(\frac{3}{2}, \infty)$
f''	+	X	+	0	−
f	∪	X	∪	infl. pont.	∩

$\boxed{2p}$

$$f\left(\tfrac{3}{2}\right) = e^{\frac{1}{1-\frac{3}{2}}} = e^{-2}$$



$\boxed{2p}$

4/

$$= \frac{1}{2}\int_0^1 y \sin y \, dy = \frac{1}{2}\left\{ \underbrace{\left[-y\cos y\right]_0^1}_{-\cos 1} + \underbrace{\int_0^1 \cos y \, dy}_{[\sin y]_0^1 = \sin 1} \right\} = \frac{1}{2}(\sin 1 - \cos$$

per partes

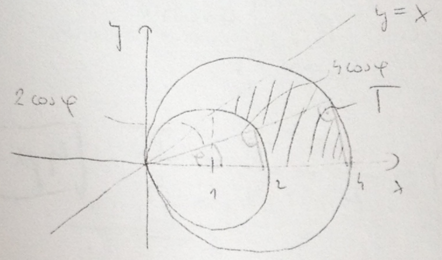
$u = y \quad u' = 1$

$v' = \sin y \quad v = -\cos y$

$\boxed{3p}$

d)

$$\iint_T xy \, dx \, dy =$$



polár. souř.: $x = r\cos\varphi$

$y = r\sin\varphi$

$|J| = r$

T: $2\cos\varphi \le r \le 4\cos\varphi$

$0 \le \varphi \le \frac{\pi}{4}$

$\boxed{4p}$

$$= \int_0^{\pi/4} \underbrace{\int_{2\cos\varphi}^{4\cos\varphi} r\cos\varphi \, r\sin\varphi \, r \, dr}_{} \, d\varphi = 60 \int_0^{\pi/4} \cos^4\varphi \sin\varphi \, d\varphi \boxed{=}$$

$\boxed{2p}$

$$\cos\varphi \sin\varphi \left[\frac{r^4}{4}\right]_{2\cos\varphi}^{4\cos\varphi} = \cos\varphi\sin\varphi\left(4^3\cos^3\varphi - 4\cos^3\varphi\right) = 60\cos^4\varphi \sin\varphi$$

$$\boxed{=}\ 60\left[-\frac{\cos^5\varphi}{5}\right]_0^{\pi/4} = -\frac{60}{5}\left[\left(\frac{\sqrt{2}}{2}\right)^5 - 1\right] = 12\left(1 - \frac{4\sqrt{2}}{32}\right) = 12\left(1 - \frac{\sqrt{2}}{8}\right)$$

$\boxed{2p}$

6)

(4) a) $z \in \mathbb{C}$ $\boxed{z^2=\bar{z}}$

algebraický tvar $z = x+iy \rightsquigarrow z^2 = x^2-y^2+2ixy$

$\bar{z} = x-iy$

$x^2-y^2+2ixy = x-iy$ $\boxed{1p}$

$\iff x^2-y^2 = x$ a $2xy=-y$

$\Downarrow$

$y(2x+1)=0 \to y=0$ nebo $\to x=-\frac{1}{2}$

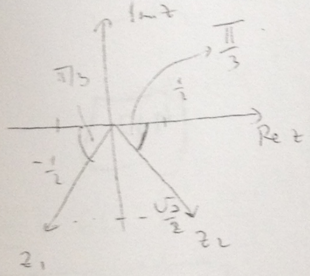
- $y=0 \Rightarrow x^2=x$

$x(x-1)=0 \to x=0$, $\to x=1$ $\boxed{\begin{matrix} z_1=0 \\ z_2=1 \end{matrix}}$ $\boxed{2p}$

- $x=-\frac{1}{2} \Rightarrow \frac{1}{4}-y^2=-\frac{1}{2}$

$y^2=\frac{3}{4} \to y=\frac{\sqrt{3}}{2}$, $\to y=-\frac{\sqrt{3}}{2}$ $\boxed{\begin{matrix} z_3=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \\ z_4=-\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{matrix}}$ $\boxed{2p}$

b)



$z_1 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i \Rightarrow z_1 = 1e^{+i\left(\frac{\pi}{3}+\pi\right)} = e^{i\frac{4\pi}{3}}$ $\boxed{2p}$

$z_2 = \frac{1}{2}-\frac{\sqrt{3}}{2}i \Rightarrow z_2 = 1e^{-i\frac{\pi}{3}} = e^{-i\frac{\pi}{3}}$ $\boxed{2p}$

$z_1^{2016}+z_2^{2016} = \left(e^{i\frac{4\pi}{3}}\right)^{2016} + \left(e^{-i\frac{\pi}{3}}\right)^{2016} =$

$= e^{i\frac{4\cdot 2016\pi}{3}} + e^{-i\frac{2016\pi}{3}} = \underbrace{e^{i4\cdot 872\pi}}_{=1} + \underbrace{e^{-i872\pi}}_{=1} = \underline{\underline{2}}$ $\boxed{2p}$

$\boxed{2p}$

$\frac{2016}{3} = 872$

5

$$\underline{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$$\det(\underline{A} - \lambda \underline{I}) = \det \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} = (1-\lambda)\overbrace{\left[(2-\lambda)(3-\lambda)-2\right]}^{\lambda^2-5\lambda+4} - 1\cdot 2 +$$

$$+ 2\cdot(2-\lambda) =$$

$$= (1-\lambda)(\lambda^2-5\lambda+4) + 2 - 2\lambda = (1-\lambda)\left[\lambda^2-5\lambda+4+2\right] =$$

$$= (1-\lambda)(\lambda-2)(\lambda-3) = 0 \implies \boxed{\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{matrix}} \qquad \boxed{6p}$$

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \implies \underline{A}\underline{v} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 + v_2 + 2v_3 \\ 2v_2 + 2v_3 \\ -v_1 + v_2 + 3v_3 \end{pmatrix}$$

- $\underline{\lambda_1 = 1} \implies$

$$\left.\begin{matrix} v_1 + v_2 + 2v_3 = v_1 \\ 2v_2 + 2v_3 = v_2 \end{matrix}\right\} \rightarrow v_2 = -2v_3$$

$$-v_1 + v_2 + 3v_3 = v_3 \qquad \rightarrow v_1 = v_2 + 2v_3 = \cancel{-4v_3}\ 0$$

$$\cancel{\underline{v} = \begin{pmatrix} -4v_3 \\ -2v_3 \\ v_3 \end{pmatrix}} = v_3 \cancel{\begin{pmatrix} -4 \\ -2 \\ 1 \end{pmatrix}} \qquad \boxed{\underline{v}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}} \qquad \boxed{2p}$$

- $\underline{\lambda_2 = 2}$

$$\begin{matrix} v_1 + v_2 + 2v_3 = 2v_1 \\ 2v_2 + 2v_3 = 2v_2 \\ -v_1 + v_2 + 3v_3 = v_3 \end{matrix} \implies \begin{matrix} v_3 = 0 \\ v_1 = v_2 \end{matrix} \qquad \boxed{\underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \qquad \boxed{2p}$$

- $\underline{\lambda_3 = 3}$

$$\begin{matrix} v_1 + v_2 + 2v_3 = 3v_1 \\ 2v_2 + 2v_3 = 3v_2 \\ -v_1 + v_2 + 3v_3 = 3v_3 \end{matrix} \implies v_1 = v_2 \qquad \rightarrow v_1 = 2v_3 \qquad \boxed{\underline{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}} \qquad \boxed{2p}$$

$$\det \underline{A} = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 1 \cdot 2 \cdot 3 = \boxed{6} \qquad \boxed{3p}$$

3

$$\left(\begin{array}{cccc|c} 1 & -3 & 0 & -6 & 9 \\ 2 & 1 & -5 & 1 & 8 \\ 0 & 2 & -1 & 2 & -5 \\ 1 & 4 & -7 & a & b \end{array}\right) \underset{\substack{S_2-2S_1 \\ S_4-S_1}}{\sim} \left(\begin{array}{cccc|c} 1 & -3 & 0 & -6 & 9 \\ 0 & 7 & -5 & 13 & -10 \\ 0 & 2 & -1 & 2 & -5 \\ 0 & 7 & -7 & a+6 & b-9 \end{array}\right) \underset{\substack{S_4-S_2, \\ 2\cdot S_2,\ 7\cdot S_3}}{\sim}$$

$$\left(\begin{array}{cccc|c} 1 & -3 & 0 & -6 & 9 \\ 0 & 14 & -10 & 26 & -20 \\ 0 & 14 & -7 & 14 & -35 \\ 0 & 0 & -2 & a-7 & b+1 \end{array}\right) \underset{S_3-S_2}{\sim} \left(\begin{array}{cccc|c} 1 & -3 & 0 & -6 & 9 \\ 0 & 14 & -10 & 26 & -20 \\ 0 & 0 & 3 & -12 & -15 \\ 0 & 0 & -2 & a-7 & b+1 \end{array}\right) \underset{\substack{\frac{S_2}{2},\ \frac{S_3}{3}=\tilde{S}_3 \\ \rightsquigarrow S_4+2\tilde{S}_3}}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & -3 & 0 & -6 & 9 \\ 0 & 7 & -5 & 13 & -10 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & a-15 & b-9 \end{array}\right) \rightsquigarrow$$

utolsó egyenlet

$$(a-15)x_4 = b-9$$

$\boxed{10p}$

1) <u>$\nexists$ megoldás</u>, ha $a-15=0$ és $b-9\neq 0$

azaz $\boxed{a=15 \text{ és } b\neq 9}$ $\boxed{4p}$

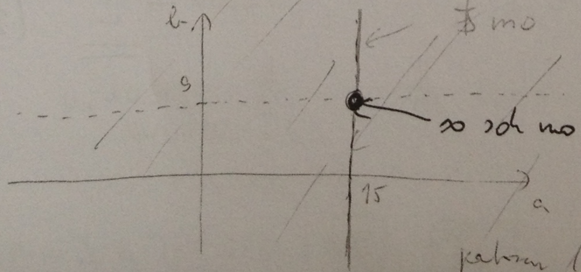
2) <u>∞ sok megoldás 1 szabad paraméterrel</u>, ha

$a-15=0$ és $b-9=0$, azaz $\boxed{a=15,\ b=9}$

$\boxed{3p}$

3) ha $a-15\neq 0$, azaz $\boxed{a\neq 15}$ $\Rightarrow$ pontosan 1 megoldás

$\boxed{3p}$



pontosan 1 mo.