1, a, $\boxed{5}$ $a_n = \left(\frac{3n+2}{3(n-1)}\right)^n = \left(\frac{3n+2}{3n-3}\right)^n = \frac{\left(1+\frac{2/3}{n}\right)^n}{\left(1-\frac{1}{n}\right)^n} \longrightarrow \frac{e^{2/3}}{e^{-1}} = e^{2/3+1} = \underline{\underline{e^{5/3}}}$

b, $\boxed{6}$ $b_n = a_n^{\,n}$. $a_n \longrightarrow e^{5/3} > 2$, tehát ha $n > N_0$, akkor

$a_n \geq 2$, így $b_n \geq 2^n \longrightarrow \infty$, tehát $b_n \longrightarrow \infty$.

2, a, $\boxed{5}$ $\int \frac{1}{\sqrt{x^2+4x+8}}\,dx = \int \frac{1}{\sqrt{(x+2)^2+4}}\,dx = \frac{1}{2}\int \frac{dx}{\sqrt{1+\left(\frac{x+2}{2}\right)^2}} \;(3)\; = \underline{\underline{\cancel{\tfrac{1}{2}}\operatorname{arsh}\left(\frac{x+2}{2}\right)\cdot\cancel{2}+C}}$ (2)

b, $\boxed{5}$ $\int \operatorname{arctg} x\,dx = \int \underbrace{1}_{u'}\cdot \underbrace{\operatorname{arctg} x}_{v}\,dx = x\operatorname{arctg} x - \int \frac{x}{1+x^2}\,dx \;(3)\; = \underline{\underline{x\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2)+C}}$ (2)

$u = x \quad v' = \frac{1}{1+x^2}$

c, $\boxed{5}$ $\int_0^4 e^{|2x-6|}\,dx = \int_0^3 e^{6-2x}\,dx + \int_3^4 e^{2x-6}\,dx \;(3)\; = \frac{-1}{2}\left[e^{6-2x}\right]_0^3 + \frac{1}{2}\left[e^{2x-6}\right]_3^4 =$

$= \frac{-1}{2}\left(1-e^6\right) + \frac{1}{2}\left(e^{+2}-1\right) = \underline{\underline{\frac{e^6+e^{+2}}{2}-1}}$ (2)

3, $\ln y + y = x+1$ (*)

a, $\boxed{5}$ $f(y) = \ln y + y$ folytonos, (1) és $\lim\limits_{y\to 0+} f(y) = -\infty$, (1) $\lim\limits_{y\to +\infty} f(y) = +\infty$, (1)

$f'(y) = \frac{1}{y}+1 > 0$, ha $y \in (0,\infty)$. Tehát f szigorúan monoton növő. (1)

A Bolzano-tétel értelmében $f(y)$ minden valós értéket felvesz, ha $y \in (0,\infty)$, és a szig. monotonitás (vagy Rolle-tétel) miatt minden értéket csak egyszer. (1)

b, $\boxed{5}$ Látható, így $x=0$, $y=1$ megoldás, tehát $y(0)=1$. (1)

(*) Deriválásával: $\frac{1}{y}y' + y' = 1 \Longrightarrow y'(0) = \frac{1}{\left(1+\frac{1}{y(0)}\right)} = \frac{1}{2}$ (2)

$\frac{-1}{y^2}y'^2 + \frac{1}{y}y'' + y'' = 0 \Longrightarrow -\frac{1}{4} + 2y''(0) = 0 \Longrightarrow y''(0) = \frac{1}{8}$ (2)

c, $\boxed{3}$ $T_2(x) = y(0) + y'(0)\cdot x + \frac{1}{2}y''(0)\,x^2 = \underline{\underline{1+\frac{1}{2}x+\frac{1}{16}x^2}}$ (3)

4, $\boxed{11}$ (H) $y' = \frac{4y}{x} \Rightarrow \int \frac{dy}{y} = 4\int \frac{dx}{x} \Rightarrow \ln|y| = 4\ln|x| + c$

$y_{H,ált}(x) = K x^4$ ④

$y_{I,p}(x) = K(x)\cdot x^4$, ② Beírva: $K'(x)x^4 + \cancel{4K(x)x^3} = \cancel{\frac{4K(x)x^4}{x}} + \frac{x^5}{x^2+3}$

$K'(x) = \frac{x}{x^2+3}$ ① ; $K(x) = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx = \frac{1}{2}\ln(x^2+3)$ ②

$y_{I,p}(x) = \frac{x^4}{2}\ln(x^2+3)$ ① ; $y_{I,ált} = y_{H,ált} + y_{I,p} = K x^4 + \frac{x^4}{2}\ln(x^2+3)$ ①

— o —

5, $\boxed{10}$ $\left.\begin{array}{l} -\cos(2x)e^x \Rightarrow \lambda_{1,2} = 1 \pm 2i \\ 7e^x \Rightarrow \lambda_3 = 1 \end{array}\right\}$ ③ $(\lambda-1+2i)(\lambda-1-2i)(\lambda-1) = ((\lambda-1)^2+4)(\lambda-1) =$

$= (\lambda^2 - 2\lambda + 5)(\lambda-1) = \lambda^3 - 3\lambda^2 + 7\lambda - 5$ ③

$y''' - 3y'' + 7y' - 5y = 0$ ①

$y_{ált} = A\sin(2x)e^x + B\cos(2x)e^x + Ce^x$; $A, B, C \in \mathbb{R}$ ③

— o —

6, $\boxed{8}$ ③ T.: Legyen $a_n > 0$.

i, Ha $\forall n \in \mathbb{N}_+$ esetén $\frac{a_{n+1}}{a_n} \le q < 1$, akkor $\sum_{n=1}^{\infty} a_n < \infty$ (konv.)

ii, Ha $\forall n \in \mathbb{N}_+$ esetén $\frac{a_{n+1}}{a_n} \ge 1$, akkor $\sum_{n=1}^{\infty} a_n = \infty$ (div.)

B.: i, $a_{n+1} \le q\, a_n \Rightarrow a_n \le a_1 \cdot q^{n-1} \Rightarrow S_n = \sum_{k=1}^{n} a_k \le a_1 \sum_{k=1}^{n} q^{k-1} =$

$= a_1 \frac{q^n - 1}{q-1} \to a_1 \cdot \frac{1}{1-q} < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ konv. ③

ii, $a_1 \le a_2 \le a_3 \le \dots \Rightarrow S_n = \sum_{k=1}^{n} a_k \ge n \cdot a_1 \xrightarrow{n\to\infty} \infty \Rightarrow S_n \to \infty$ div. ②

— 3 — Szig

7* / [8]

$f(x,y) = xy + y^2 - y + x^2 - 6$

$\operatorname{grad} f(x,y) = \begin{bmatrix} y+2x \\ x+2y-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} y+2x=0 \Rightarrow y=-2x \\ x+2y-1=0 \end{matrix}\right\}$ (2)

$x - 4x = 1;\ \underline{x_0 = -\frac{1}{3};\ y_0 = \frac{2}{3}}$ (2)

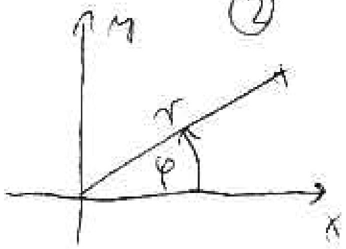
$|\underline{\underline{H}}| = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1 = 3 > 0$ (2)

$f''_{xx}(x_0,y_0) = 2 > 0$

} f-nek lokális minimuma van $\left(-\frac{1}{3}, \frac{2}{3}\right)$-ban. (2)

— o —

8* / [6] a)

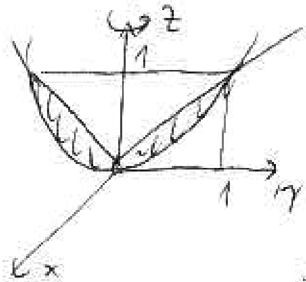


(2)

$x = r\cos\varphi$ (2)
$y = r\sin\varphi$

$|\underline{\underline{J}}| = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$ (2)

[9] b) Henger-koordinátákkal: (3)



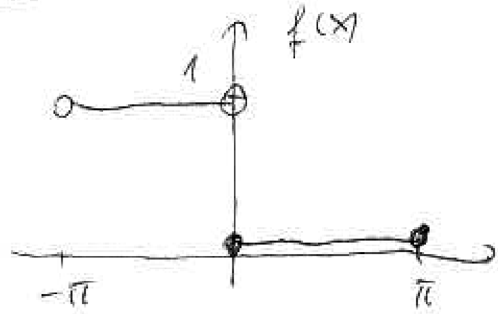
$0 \le r \le 1$ (3)
$0 \le \varphi \le 2\pi$
$r^2 \le z \le r$

$V = \int_{r=0}^{1} \int_{\varphi=0}^{2\pi} \int_{z=r^2}^{r} r \, dz\, d\varphi\, dr =$

$= 2\pi \int_{r=0}^{1} r\,(r - r^2)\, dr = 2\pi\left(\frac{1}{3} - \frac{1}{4}\right) = \underline{\underline{\frac{\pi}{6}}}$ (3)

— o —

9* / [9]



$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} 1\, dx = 1$ (1)

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \cos(nx)\, dx =$

$= \frac{1}{n\pi} \left[\sin(nx)\right]_{-\pi}^{0} = 0$ (3)

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx = \frac{1}{\pi} \int_{-\pi}^{0} \sin(nx)\, dx = \frac{1}{\pi} \left[\frac{-\cos(nx)}{n}\right]_{-\pi}^{0} =$

$= \frac{1}{n\pi}\left(-1 + (-1)^n\right) = \begin{cases} \frac{-2}{n\pi}, & \text{ha } n \text{ páratlan} \\ 0, & \text{ha } n \text{ páros} \end{cases}$ (3)

$\underline{\underline{\Phi(x) = \frac{1}{2} + \frac{-2}{\pi}\sin x - \frac{2}{3\pi}\sin(3x) - \frac{2}{5\pi}\sin(5x) + \dots}}$ (2)