str. 1. VD1. Mintamegoldás 2013. 12. 23.

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1, a,

$\boxed{6}$ $\left(\frac{2}{3}-\varepsilon\right)^n < a_n = \left(\frac{2n+2}{3n+4}\right)^n < \left(\frac{2}{3}+\varepsilon\right)^n$, mert $\frac{2n+2}{3n+4} = \frac{2+2/n}{3+4/n} \longrightarrow \frac{2}{3}$

$\left(\frac{2}{3}-\varepsilon\right)^n \xrightarrow{n\to\infty} 0$, ha $\varepsilon < \frac{1}{3}$: $\left(\frac{2}{3}+\varepsilon\right)^n \xrightarrow{n\to\infty} 0$

Tehát a rendőr elv értelmében $\underline{\underline{\lim\limits_{n\to\infty} a_n = 0.}}$

$\boxed{5}$ $b_n = \left(\frac{5n+2}{5n+4}\right)^n = \left(\frac{1+\frac{2}{5n}}{1+\frac{4}{5n}}\right)^n = \frac{\left(1+\frac{2/5}{n}\right)^n}{\left(1+\frac{4/5}{n}\right)^n}$ 3 $\xrightarrow{n\to\infty} \frac{e^{2/5}}{e^{4/5}} = \underline{\underline{e^{-2/5}}}$ 2

$\boxed{4}$ b, A majoráns kritériummal igazoljuk, hogy $\sum\limits_{n=1}^{\infty} a_n$ konvergens.
Láttuk, hogy ha $n > N(\varepsilon)$ akkor $a_n < \left(\frac{2}{3}+\varepsilon\right)^n$. Legyen $\varepsilon = \frac{1}{6}$.
Tehát $a_n < \left(\frac{2}{3}+\frac{1}{6}\right)^n = \left(\frac{5}{6}\right)^n$, és $\sum\limits_{n=1}^{\infty}\left(\frac{5}{6}\right)^n < \infty$, mert $\left|\frac{5}{6}\right| < 1$

3, a,

$\boxed{2}$ $\lim\limits_{x\to 0+} \frac{\ln(x)}{x} = \frac{-\infty}{0+} = -\infty$ 2 (nem határozatlan alak)

$\boxed{3}$ b, $\lim\limits_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim\limits_{x\to\infty} \frac{1/x}{1} = 0$ 3

c,

$\boxed{4}$ $\lim\limits_{x\to\infty} \sqrt[x]{x} = \lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\frac{1}{x}\ln x}$ 2 $= e^{\lim\limits_{x\to\infty}\frac{\ln x}{x}} = e^0 = 1$ 2

$\boxed{2}$ d, $g'(x) = \left(e^{\frac{\ln x}{x}}\right)' = \left(\frac{\ln x}{x}\right)' e^{\frac{\ln x}{x}} = \frac{\frac{1}{x}\cdot x - \ln x\cdot 1}{x^2}\cdot\sqrt[x]{x} = \underline{\underline{\frac{1-\ln x}{x^2}\cdot\sqrt[x]{x}}}$

2, a, $\boxed{7}$

$$\lim_{x\to 0}\frac{e^{2x^2}-1}{\ln(1+3x^2)} \underset{\text{"}\frac{0}{0}\text{" alak}}{\overset{L'H}{=}} \lim_{x\to 0}\frac{4\cancel{x}\,e^{2x^2}}{\left(\frac{6\cancel{x}}{1+3x^2}\right)} \ (3) = \lim_{x\to 0}\frac{2}{3}e^{2x^2}(1+3x^2) = \ (3)$$

$$= \underline{\underline{\frac{2}{3}}} \ (1)$$

$\boxed{5}$ b,

$$\lim_{x\to\infty}\frac{\operatorname{sh}(x+3)}{\operatorname{ch}(x+2)} = \lim_{x\to\infty}\frac{\frac{1}{2}\left(e^{x+3}-e^{-x-3}\right)}{\frac{1}{2}\left(e^{x+2}+e^{-x-2}\right)} \ (2) = \lim_{x\to\infty}\frac{e^3-\overbrace{e^{-2x-3}}^{\to 0}}{e^2+\underbrace{e^{-2x-2}}_{\to 0}} \ (2) =$$

$$= \frac{e^3}{e^2} = \underline{\underline{e}} \ (1)$$

4, a, $\boxed{3}$

$$\left(f^{-1}\right)' = \frac{1}{f'\circ f^{-1}}, \quad \text{vagy} \quad \left(f^{-1}(x)\right)' = \frac{1}{f'(f^{-1}(x))}$$

b, $\boxed{7}$ $x \geq 0$

$$(\operatorname{ar\,ch} x)' = \frac{1}{\operatorname{ch}'(\operatorname{ar\,ch} x)} = \frac{1}{\operatorname{sh}(\underbrace{\operatorname{ar\,ch} x})} \ (2) = \frac{1}{\operatorname{sh} y} = \frac{1}{\sqrt{\operatorname{ch}^2 y - 1}} \ (2) =$$

$$= \frac{1}{\sqrt{x^2-1}} \ (1)$$

$$\left|\begin{array}{c} y = \operatorname{ar\,ch} x \\ \operatorname{ch} y = x \\ \operatorname{ch}^2 y - \operatorname{sh}^2 y = 1 \end{array}\right\} \ (2) \Rightarrow \operatorname{sh} y = \sqrt{\operatorname{ch}^2 y - 1} = \sqrt{x^2-1}$$

5, a, $\boxed{3}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

b, (3) $\underline{\text{T.:}}$ Ha $x_0 \in \operatorname{int} D_f$, f diff.-ható x_0-ban, és f-nek x_0-ban lokális szélsőértéke van, akkor $f'(x_0)=0$.

(6) → $\underline{\text{B.:}}$ Pl. legyen f-nek x_0-ban lokális maximuma.

$$f'_+(x_0) = \lim_{\varepsilon\to 0+}\frac{\overbrace{f(x_0+\varepsilon)-f(x_0)}^{\leq 0}}{\underbrace{\varepsilon}_{>0}} \leq 0$$

$$f'_-(x_0) = \lim_{\varepsilon \to 0-} \frac{\overbrace{f(x_0+\varepsilon) - f(x_0)}^{\leq 0}}{\underbrace{\varepsilon}_{<0}} \geq 0$$

Mivel f diff. ható, $f'_-(x_0) = f'_+(x_0)$, ami csak úgy lehet, hogy $f'(x_0) = f'_-(x_0) = f'_+(x_0) = 0$.

Hasonló a bizonyítás lokális minimumra is.

6* a, Parciálisan integrálunk. (4)

$\boxed{7}$ $$\int \underbrace{(3x+2)}_{u} \underbrace{\sin(5x)}_{v'}\, dx = -\frac{1}{5}(3x+2)\cos(5x) - \left(-\frac{3}{5}\right)\int \cos(5x)\, dx =$$ (3)

$$u' = 3 \quad v = \frac{-\cos(5x)}{5} \quad \Big| = \frac{-3x-2}{5}\cos(5x) + \frac{3}{5}\cdot\frac{1}{5}\sin(5x) + C$$

$\boxed{13}$ b, $$I = \int \frac{e^x}{e^{2x} + 5e^x + 4}\, dx \; ; \quad t = e^x ; \; x = \ln t ; \; dx = \frac{1}{t}\, dt$$

$$\Rightarrow \int \frac{\cancel{t}}{t^2+5t+4} \cdot \frac{\cancel{1}}{\cancel{t}}\, dt$$ (3) racionális tört függvény

$t^2 + 5t + 4 = (t+4)(t+1)$, tehát a parciális törtek:

$$\frac{1}{t^2+5t+4} = \frac{A}{t+4} + \frac{B}{t+1} \Rightarrow 1 = A(t+1) + B(t+4)$$ (3)

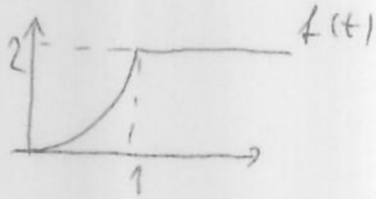
$$\left.\begin{aligned} t=-1:&\; 1 = 3B \Rightarrow B = \frac{1}{3} \\ t=-4:&\; 1 = -3A \Rightarrow A = -\frac{1}{3} \end{aligned}\right\}$$ (3)

$$\int \frac{1}{t^2+5t+4}\, dt = \int \frac{-1/3}{t+4}\, dt + \int \frac{1/3}{t+1}\, dt = \frac{-1}{3}\ln|t+4| + \frac{1}{3}\ln|t+1| + C$$ (3)

$$\Rightarrow I = -\frac{1}{3}\ln(e^x+4) + \frac{1}{3}\ln(e^x+1) + C$$ (1)

7* $\boxed{9}$

$$f(t)=\begin{cases} 2t^3, & \text{ha } t\in[0,1] \\ 2, & \text{ha } t>1 \end{cases}$$



Ha $0\le x\le 1$, akkor $F(x)=\int_0^x 2t^3\,dt = 2\cdot\left[\frac{t^4}{4}\right]_0^x = \underline{\underline{\frac{x^4}{2}}}$ (2)

Ha $1\le x$, akkor

$$F(x)=\int_0^1 2t^3\,dt+\int_1^x 2\,dt=\underline{\underline{\frac{1}{2}+2(x-1)}}$$ (4)

(3) Mivel f folytonos, ezért az integrálszámítás II. alaptétele értelmében F deriválható, és $F'(x)=f(x)$ $(x>0)$.

8* a, $\boxed{5}$

$$\int \frac{1}{4x^2+4x+10}\,dx=\int\frac{1}{(2x+1)^2+9}\,dx=\frac{1}{9}\int\frac{1}{1+\left(\frac{2x+1}{3}\right)^2}\,dx=$$ (3)

$$=\frac{1}{9}\cdot \operatorname{arctg}\left(\frac{2x+1}{3}\right)\cdot\frac{3}{2}+C=\underline{\underline{\frac{1}{6}\operatorname{arctg}\left(\frac{2x+1}{3}\right)+C}}$$ (2)

b, $\boxed{6}$

$$\int_0^\pi \cos^2(x)\,dx=\int_0^\pi\frac{1+\cos(2x)}{2}\,dx=\int_0^\pi\frac{1}{2}\,dx+\frac{1}{2}\int_0^\pi\cos(2x)\,dx=$$ (2)

$$=\frac{\pi}{2}+\frac{1}{2}\left[\frac{\sin(2x)}{2}\right]_0^\pi=\frac{\pi}{2}+\frac{1}{2}\left(\frac{0}{2}-\frac{0}{2}\right)=\underline{\underline{\frac{\pi}{2}}}$$ (2) (2)