1, $\boxed{13}$ $\quad y'(x) = \dfrac{(y(x)+2)^2}{1+x^2}$; $\quad y(1) = -1$

$$\int \frac{dy}{(2+y)^2} = \int \frac{dx}{1+x^2} \quad 3$$

$$\frac{-1}{2+y} \; 3 = \operatorname{arctg} x \; 3 + C \quad ; \quad \frac{-1}{2-1} = \operatorname{arctg} 1 + C$$

$$C = -1 - \frac{\pi}{4} \quad 2$$

$$y(x) = \frac{1}{1 + \frac{\pi}{4} - \operatorname{arctg} x} - 2 \quad 2$$

— o —

2, $\boxed{14}$ $\quad y' = 6xy + 3x$

a, Szeparálhatóként:

$$\frac{dy}{dx} = 3x(2y+1) \qquad y \equiv -\frac{1}{2} \text{ megoldás}$$

$$\int \frac{dy}{2y+1} = \int 3x \, dx \quad 3$$

$$\frac{1}{2} \ln|2y+1| \; 3 = \frac{3}{2} x^2 + C \; 3 \quad ; \; C \in \mathbb{R}$$

$$|2y+1| = K \cdot e^{3x^2} \quad ; \; K > 0$$

$$y = -\frac{1}{2} \pm K e^{3x^2} \; 3 \longrightarrow y_{ált}(x) = -\frac{1}{2} + \tilde{K} e^{3x^2} \quad 2$$

$$\tilde{K} \in \mathbb{R}$$

b, Lineárisként:

(H): $y' = 6xy$; $\displaystyle\int \frac{dy}{y} = \int 6x \, dx$; $\ln|y| = 3x^2 + C$;

$$y_{H,ált}(x) = K \cdot e^{3x^2} \; ; \; K \in \mathbb{R} \quad 5$$

(I): $y_{I,part}(x) = K(x) \cdot e^{3x^2}$; $y'_{I,p}(x) = (K'(x) + 6x K(x))\, e^{3x^2}$

$$(K'(x) + \cancel{6x K(x)})\, e^{3x^2} = 3x\,(\cancel{2K(x)e^{3x^2}} + 1)$$

$K'(x) = 3x e^{-3x^2}$; $K(x) = \displaystyle\int 3x e^{-3x^2} dx = -\frac{1}{2} e^{-3x^2}$; $y_{I,p}(x) = -\dfrac{1}{2}$ 5

$$y_{I,ált} = y_{H,ált} + y_{I,p} = -\frac{1}{2} + K e^{3x^2} \quad 2$$

3, $\boxed{14}$

$$y' = \frac{-2x+y}{y-x}\ ; \quad x>0\ ; \quad u(x) = \frac{y(x)}{x}$$

$$y = u\cdot x\ ;\quad y' = u'\cdot x + u \quad (2)$$

$$u'(x)\cdot x + u(x) = \frac{-2+u(x)}{u(x)-1} \ (2)\ ; \quad u'(x)\cdot x = \frac{-u^2(x)+2u(x)-2}{u(x)-1}$$

$$-\int \frac{u-1}{u^2-2u+2}\,du = \int \frac{dx}{x} \ (2)\ ; \quad -\frac{1}{2}\ln\left|u^2-2u+2\right| \ (2) = \ln x + C \ (2)$$

$\frac{f'}{f}$ alak

$$\frac{1}{\sqrt{u^2-2u+2}} = Kx \ (2)\ ; \quad K\in\mathbb{R}\ ; \quad \underline{\underline{\frac{1}{\sqrt{y^2-2xy+x^2}} = Kx^2}} \ (2)$$

— o —

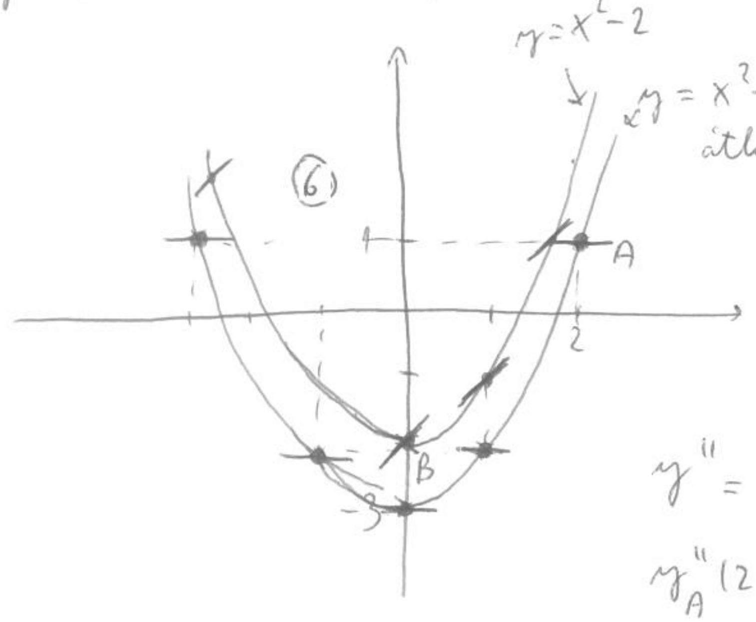
4, $\boxed{18}$ $\quad y' = y - x^2 + 3\ ;\quad A(2,1)\ ;\quad B(0,-2)$

a, Az A-pontban:

$y'(A) = 1-2^2+3 = 0$; izoklína: $y - x^2 + 3 = 0$; $y = x^2-3$ (3)

A B pontban:

$y'(B) = -2-0^2+3 = 1$; izoklína: $y - x^2 + 3 = 1$; $y = x^2-2$ (2)



b, Legyen $y_A(x)$ az A-n áthaladó megoldás.

$y_A'(2) = 1-2^2+3 = 0$

$y'' = y' - 2x$

$y_A''(2) = y_A'(2) - 4 = -4$ (4)

c, Mivel $y_A'(2) = 0$, és $y_A''(2) = -4 < 0$, ezért itt a megoldásnak lokális maximuma van. (3)

5, $\boxed{14}$ $y'' - 4y' + 4y = e^{2x}$

(H) $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0 \;;\; \lambda = 2$ ②

$y_{H,\text{ált}}(x) = C_1 e^{2x} + C_2 x e^{2x}$ (Belső rezonancia) ③

(I) $y_{I,p}(x) = A x^2 e^{2x}$ (Külső rezonancia) ②

$$\left.\begin{aligned} y'_{I,p}(x) &= (2Ax + 2Ax^2)e^{2x} \\ y''_{I,p}(x) &= (2A + 4Ax + 4Ax + 4Ax^2)e^{2x} = (2A + 8Ax + 4Ax^2)e^{2x} \end{aligned}\right\} ③$$

Visszaírva:

$$(2A + \cancel{8Ax} + \cancel{4Ax^2})e^{2x} - 4(\cancel{2Ax} + \cancel{2Ax^2})e^{2x} + \cancel{4Ax^2}e^{2x} = e^{2x}$$

$$2A = 1 \;;\; A = \frac{1}{2} \;;\; y_{I,p}(x) = \frac{x^2}{2} e^{2x} \quad ②$$

$$y_{I,\text{ált}}(x) = y_{H,\text{ált}}(x) + y_{I,p}(x) = \underline{\underline{\left(C_1 + C_2 x + \frac{x^2}{2}\right)e^{2x}}} \quad ②$$

— o —

6, $\boxed{15}$ a, $\displaystyle\sum_{n=1}^{\infty} \underbrace{\left(\frac{2n+5}{3n+1}\right)^{n^2}}_{a_n > 0}$; $\displaystyle\sqrt[n]{a_n} = \left(\frac{2n}{3n}\right)^n \left(\frac{1 + \frac{5}{2n}}{1 + \frac{3}{2n}}\right)^n = \underbrace{\left(\frac{2}{3}\right)^n}_{\downarrow\, 0} \cdot \underbrace{\frac{\left(1 + \frac{5/2}{n}\right)^n}{\left(1 + \frac{3/2}{n}\right)^n}}_{\downarrow\, e^{5/2 - 3/2}} \to 0 < 1$ ⑥

Tehát a sor konvergens.

b, $\displaystyle\sum_{n=1}^{\infty} \underbrace{\left(\frac{2n+5}{2n+1}\right)^{n^2}}_{a_n > 0}$; $\displaystyle\sqrt[n]{a_n} = \frac{\left(1 + \frac{5/2}{n}\right)^n}{\left(1 + \frac{1/2}{n}\right)^n} \longrightarrow e^{5/2 - 1/2} = e^{3/2} > 1$ ⑤

Tehát a sor divergens.

c, $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{3^n}{(2n+1)!}}_{a_n > 0}$; $\displaystyle\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{3^n} = \frac{3}{(2n+2)(2n+1)} \xrightarrow{n \to \infty} 0 < 1$ ④

Tehát a sor konvergens.

7, $\boxed{12}$ Fibonacci - sorozat: $f_{n+1} = f_n + f_{n-1}$ (2)

$$q^2 = q + 1 \; ; \; q^2 - q - 1 = 0 \; ; \; q_{12} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 1} = \frac{1 \pm \sqrt{5}}{2} \quad (2)$$

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \; (2); \qquad \left.\begin{aligned} f_0 &= A + B = 1 \\ f_1 &= A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{aligned}\right\} (2)$$

$$A = 1 - B, \quad (1-B)\frac{1+\sqrt{5}}{2} + B\,\frac{1-\sqrt{5}}{2} = 1 \; ; \; \frac{1+\sqrt{5}}{2} - \sqrt{5}B = 1$$

$$B = \frac{\sqrt{5}-1}{2\sqrt{5}} = \frac{5-\sqrt{5}}{10} \; ; \quad A = \frac{\sqrt{5}+5}{10}$$

$$\underline{\underline{f_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n}} \quad (4)$$

—— ∘ —— PÓTFELADATOK —— ∘ ——

8, $\boxed{10}$ $e^x \cos(2x) \implies \lambda_1 = 1 + 2i \; ; \; \lambda_2 = 1 - 2i$ (3)

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \overset{(2)}{=} (\lambda - 1 - 2i)(\lambda - 1 + 2i) = (\lambda - 1)^2 + 4 = \lambda^2 - 2\lambda + 5 \quad (3)$$

$$\implies \underline{\underline{y'' - 2y' + 5y = 0}} \quad (2)$$

—— ∘ ——

9, $\boxed{10}$ a, Legyen $a_n > 0$ $\forall n \in \mathbb{N}$ esetén.

i, Ha $\exists \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, akkor $\sum\limits_{n\in\mathbb{N}} a_n < \infty$ (konvergens)

ii, Ha $\exists \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, akkor $\sum\limits_{n\in\mathbb{N}} a_n = \infty$ (divergens)

b, $$\sum_{n=1}^{\infty} \underbrace{\frac{(2n+1)!}{(n+2)^2}}_{a_n > 0} \; ; \quad \frac{a_{n+1}}{a_n} = \frac{(2n+3)!}{(n+3)^2} \cdot \frac{(n+2)^2}{(2n+1)!} = \frac{(2n+2)(2n+3)}{\left(\frac{n+3}{n+2}\right)^2} \xrightarrow{n\to\infty} \infty > 1$$

(5)

Tehát a sor divergens.

De ez onnan is látszik, hogy $a_n \xrightarrow{n\to\infty} \infty$.