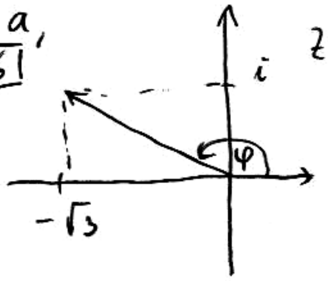
ANALÍZIS 1. 1. ZH. MEGOLDÁS | -1- | 2017. OKT. 12. | α

1, a, $\boxed{6}$

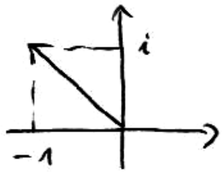


$z = -\sqrt{3} + i = \underline{\underline{2 \cdot e^{i\frac{5\pi}{6}}}}$ (1)

$|z| = \sqrt{3+1} = 2$ (2)

$\sin\varphi = \frac{1}{2}$; $\cos\varphi = -\frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{5\pi}{6}$ (3)

b, $\boxed{7}$ $\frac{i}{3-3i} = \frac{i(3+3i)}{18} = \frac{1}{6}(-1+i) \overset{(3)}{=} \underbrace{\frac{\sqrt{2}}{6}}_{(2)} \cdot e^{i\overbrace{\frac{3\pi}{4}}^{(2)}}$



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2, A z_1 pontot az origó körül $\pm 120°$-kal kell elforgatni:

$\boxed{12}$ $z_{2,3} = e^{\pm i\frac{2\pi}{3}} \cdot z_1 = \left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)(1+2i) = \underline{\underline{\left(-\frac{1}{2} \mp \sqrt{3}\right) + i\left(-1 \pm \frac{\sqrt{3}}{2}\right)}}$ (6)

(6)

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3, a, $\boxed{5}$ D.: $\lim\limits_{n\to\infty} a_n = A$, ha $\forall \varepsilon > 0$ esetén $\exists N(\varepsilon) \in \mathbb{N}$, hogy $\forall n > N(\varepsilon)$ mellett $|a_n - A| < \varepsilon$.

b, $\boxed{8}$ $|a_n - A| = \left|\frac{3n^4+n}{n^4-100} - 3\right| \overset{(2)}{=} \frac{n+300}{|n^4-100|} \underset{\text{ha } n \geq 4}{=} \frac{n+300}{n^4-100} \overset{(2)}{\underset{\text{ha } n > 300 \text{ és } \frac{n^4}{2} > 100}{<}} \frac{2n}{\frac{1}{2}n^4} =$

$= \frac{4}{n^3} \overset{(2)}{<} \varepsilon \iff n > \sqrt[3]{\frac{4}{\varepsilon}}$; $\underline{\underline{N(\varepsilon) \geq \max\left\{\sqrt[3]{\frac{4}{\varepsilon}}, 300\right\}}}$ (2)

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4, a, $\boxed{7}$ $a_n = \frac{1}{\sqrt{n^2+3n+4} - \sqrt{n^2+1}} = \frac{\sqrt{n^2+3n+4} + \sqrt{n^2+1}}{(\cancel{n^2}+3n+4) - (\cancel{n^2}+1)} \overset{(3)}{=} \frac{\sqrt{n^2+3n+4} + \sqrt{n^2+1}}{3n+3} =$

$= \frac{\sqrt{1+3/n+4/n^2} + \sqrt{1+1/n^2}}{3+3/n}$ (2) $\xrightarrow{n\to\infty} \frac{1+1}{3} = \underline{\underline{\frac{2}{3}}}$ (2) $\left(\frac{1}{n}, \frac{1}{n^2} \to 0\right)$

b, $\boxed{7}$ $b_n = \frac{n^6 + 2^{2n+3}}{6^n + n^2} = \frac{n^6 + 8\cdot 4^n}{6^n + n^2} \overset{(2)}{=} \frac{\underbrace{\left(\frac{n^6}{6^n}\right)}_{\to 0} + 8\cdot\underbrace{\left(\frac{2}{3}\right)^n}_{\to 0}}{1 + \underbrace{\frac{n^2}{6^n}}_{\to 0}}$ (3) $\longrightarrow \frac{0}{1} = \underline{\underline{0}}$ (2)

4, (folyt.)

c, $\boxed{7}$ $c_n = \left(\frac{2n+1}{3+2n}\right)^n = \frac{\left(1+\frac{1}{2n}\right)^n}{\left(1+\frac{3}{2n}\right)^n}$ ④ $\longrightarrow \frac{e^{1/2}}{e^{3/2}} = \underline{\underline{e^{-1}}}$ ③

d, Rendőr-elv:
$\boxed{9}$

$$\underbrace{\frac{2}{\sqrt[n]{3}\cdot\left(\sqrt[n]{n}\right)^2}}_{\downarrow \atop 2} \overset{④}{=} \sqrt[n]{\frac{2^n}{3n^2}} \le d_n = \sqrt[n]{\frac{2^n+n^2}{2n^2+1}} \le \sqrt[n]{\frac{2^n+2^n}{1}} = \overset{④}{\underbrace{\sqrt[n]{2}\cdot 2}_{\downarrow n\to\infty \atop 2}}$$

Tehát $\underline{\underline{d_n \to 2}}$ ①

5, $\boxed{18}$ $a_1 = 3$; $a_{n+1} = \frac{10}{7-a_n}$

a, Teljes indukcióval dolgozunk. $2 < a_1 = 3 < 5$ ✓
$\boxed{6}$ Tegyük fel, hogy $2 < a_n < 5 \overset{(-7)}{\Longrightarrow} -5 < a_n - 7 < -2 \overset{\cdot(-1)}{\Longrightarrow}$
$\Longrightarrow 5 > 7 - a_n > 2 \overset{\uparrow(-1)}{\Longrightarrow} \frac{1}{5} < \frac{1}{7-a_n} < \frac{1}{2} \overset{\cdot(10)}{\Longrightarrow} 2 < \frac{10}{7-a_n} = a_{n+1} < 5$ ✓

b, Most is teljes indukcióval dolgozunk.
$\boxed{6}$ $a_1 = 3 > a_2 = \frac{10}{7-3} = \frac{5}{2} = 2{,}5$ ✓
T.f.h. $a_n > a_{n+1} \Longrightarrow a_n - 7 > a_{n+1} - 7 \Longrightarrow 0 < 7 - a_n < 7 - a_{n+1} \Longrightarrow$
$\Longrightarrow \frac{10}{7-a_n} = a_{n+1} > \frac{10}{7-a_{n+1}} = a_{n+2}$ ✓

c, $\boxed{6}$ $A = \frac{10}{7-A} \Longrightarrow A^2 - 7A - 10 = (A-2)(A-5) = 0$

$A_1 = 2$, $A_2 = 5$ ③

Mivel a_n monoton csökken, és alulról korlátos, ezért létezik a határértéke. ② $a_1 = 3 \Longrightarrow \lim_n a_n \le 3 \Longrightarrow \underline{\underline{A = \lim_n a_n = 2}}$ ①

6, $\boxed{14}$ $a_n = \frac{2^{n+1}}{(-9)^n+3^{2n}+2^n} = \frac{2\cdot 2^n}{(-9)^n+9^n+2^n}$

Ha n páros: $a_n = \frac{2\cdot 2^n}{2\cdot 9^n+2^n} = \frac{2\cdot\left(\frac{2}{9}\right)^n}{2+\left(\frac{2}{9}\right)^n} \longrightarrow \frac{0}{2} = 0$ (4)

Ha n páratlan: $a_n = \frac{2\cdot 2^n}{\cancel{-9^n}+\cancel{9^n}+2^n} = 2$ (4)

Tehát egy konstans és egy konvergens sorozat van összefésülve.
Torlódási pontok: $S = \{0, 2\}$; $\liminf a_n = 0$ (2); $\limsup a_n = 2$ (2);
A limesz $\nexists$. (2)

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<u>IMSc</u> A konstrukcióból adódik, hogy

$$a_n = a_{10\cdot n} = a_{100\cdot n} = \dots = a_{10^k\cdot n} \quad \forall k \in \mathbb{N}\text{-re},$$

tehát az (a_n) sorozatban minden $[0.1, 1)$ intervallumba eső véges tizedes tört végtelen sokszor szerepel, tehát ezek torlódási pontok. (3) Ugyanakkor ezekkel tetszőlegesen megközelíthetünk minden $[0.1, 1]$ intervallumba eső számot, tehát az $S=[0.1, 1]$ intervallum pontjai torlódási pontok. (3)

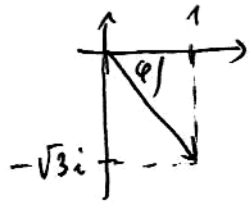
Másrészt, $\forall x \in \mathbb{R}\setminus S$ esetén $\exists \varepsilon > 0$, hogy $(x-\varepsilon, x+\varepsilon)$ és $[0.1, 1]$ diszjunktak, és $\forall n \in \mathbb{N}$-re $a_n \in [0.1, 1)$. Tehát $\mathbb{R}\setminus S$ pontjai nem torlódási pontok. (2)

Tehát a torlódási pontok: $\underline{\underline{S=[0.1, 1]}}$

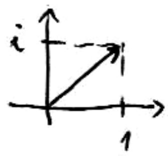
ANALÍZIS 1. ZH1. MEGOLDÁS | β | −4− | (TÖMÖR) | 2017. 10. 12.

1, a, $\boxed{6}$ $z = 1-\sqrt{3}\,i = \underline{\underline{2e^{-i\frac{\pi}{3}}}} = \underline{\underline{2e^{i\frac{5\pi}{3}}}}$ 1

$|z| = 2$ 2 ; $\varphi = -\frac{\pi}{3}$ 3



b, $\boxed{7}$ $\frac{i}{2+2i} = \frac{2+2i}{8} = \frac{1}{4}(1+i) = \underline{\underline{\frac{\sqrt{2}}{4}e^{i\frac{\pi}{4}}}}$ 3 2 2



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2, $\boxed{12}$ $z_{2,3} = e^{\pm i\frac{2\pi}{3}} z_1 = \left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)(2-i) = \underline{\underline{\left(-1 \pm \frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2} \pm \sqrt{3}\right)}}$ 6 6

— o —

3, a, $\boxed{5}$ $(\forall \varepsilon > 0)(\exists N(\varepsilon) > 0)(\forall n > N(\varepsilon)) : |a_n - A| < \varepsilon$

b, $\boxed{8}$ $\left|\frac{5n^6+n}{n^6-200} - 5\right| = \frac{n+1000}{|n^6-200|} \le \frac{2n}{\frac{1}{2}n^6} = \frac{4}{n^5} < \varepsilon$ 2 2 2

ha $n > 1000$ $\quad$ $\underline{\underline{N(\varepsilon) > \max\left\{\sqrt[5]{\frac{4}{\varepsilon}}, 1000\right\}}}$ 2

— o —

4, a, $\boxed{7}$ $a_n = \frac{\sqrt{n^2+2n+5}+\sqrt{n^2+1}}{(\cancel{n^2}+2n+5)-(\cancel{n^2}+1)} = \frac{\sqrt{1+\frac{2}{n}+\frac{5}{n^2}}+\sqrt{1+\frac{1}{n^2}}}{2+\frac{4}{n}} \longrightarrow \frac{1+1}{2} = \underline{\underline{1}}$ 3 2 2

b, $\boxed{7}$ $b_n = \frac{n^4+27\cdot 9^n}{10^n+n^3} = \frac{n^4\left(\frac{1}{10}\right)^n + 27\left(\frac{9}{10}\right)^n}{1+n^3\left(\frac{1}{10}\right)^n} \longrightarrow \frac{0}{1} = \underline{\underline{0}}$ 2 3 2

c, $\boxed{7}$ $c_n = \frac{\left(1+\frac{2}{3n}\right)^n}{\left(1+\frac{3}{3n}\right)^n} \longrightarrow \frac{e^{2/3}}{e} = \underline{\underline{e^{-1/3}}}$ 4 3

d, $\boxed{9}$ $3 \leftarrow \underbrace{\frac{3}{\sqrt[n]{4n^3}} = \sqrt[n]{\frac{3^n}{4n^3}}}_{4} \le \underset{\downarrow \atop 3\ (1)}{d_n} = \sqrt[n]{\frac{3^n+n^3}{3n^3+1}} \le \underbrace{\sqrt[n]{\frac{2\cdot 3^n}{1}} = \sqrt[n]{2}\cdot 3}_{4} \longrightarrow 3$

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5, a, $3 < a_1 = 4 < 7$

$\boxed{6}$ T.f.h. $3 < a_n < 7 \Rightarrow 7 > 10 - a_n > 3 \Rightarrow 3 < \frac{21}{10-a_n} = a_{n+1} < 7$ ✓

b, $a_1 = 4 > a_2 = \frac{21}{6} = \frac{7}{2} = 3.5$

$\boxed{6}$ T.f.h. $a_n > a_{n+1} \Rightarrow 0 < 10 - a_n < 10 - a_{n+1} \Rightarrow \frac{21}{10-a_n} = a_{n+1} < \frac{21}{10-a_{n+1}} = a_{n+2}$ ✓

c, $A = \frac{21}{10-A} \Rightarrow A^2 - 10A + 21 = (A-3)(A-7) = 0$; $A_1 = 3$ ✓ (3)

(3)

$\cancel{A_2 = 7}$ { [illegible]...

$\boxed{6}$

6, Ha n páros: $a_n = \frac{9 \cdot 3^n}{2 \cdot 4^n + 3^n} \longrightarrow 0$ (4)

$\boxed{14}$

Ha n páratlan: $a_n = \frac{9 \cdot 3^n}{3^n} = 9$ (4)

$S = \{0, 9\}$; lim inf $a_n = 0$, (2) lim sup $a_n = 9$, (2) $\nexists$ lim a_n. (2)
