Űrkommunikáció
Space Communication
2023/12.

Quaternary (Quadrature) Phase Shift Keying – QPSK

Waveforms:

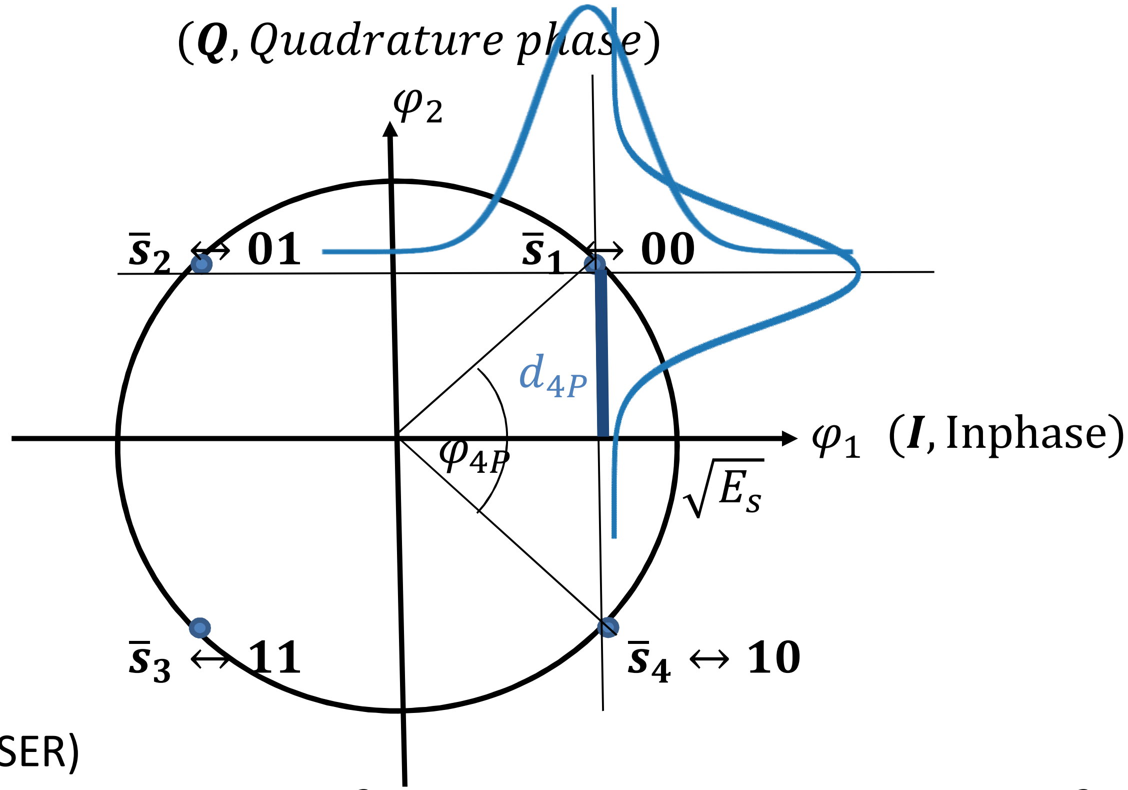
$$s_i(t) = \sqrt{\frac{2 \cdot E_s}{T_s}} \cos\left(\omega_c \cdot t + i \cdot \frac{\pi}{2} + \varphi\right)$$

$E_s = 2 \cdot E_b$ and $T_s = 2 \cdot T_b$

φ could be any, here $\frac{\pi}{4}$

$$\varphi_{4P} = \frac{\pi}{2} \quad \boldsymbol{d_{4P}} = \sqrt{E_s} \sin\frac{\pi}{4} = \sqrt{\frac{E_s}{2}}$$

Gray mapping (coding)
Small Hamming distance
To Short Euclidean distance



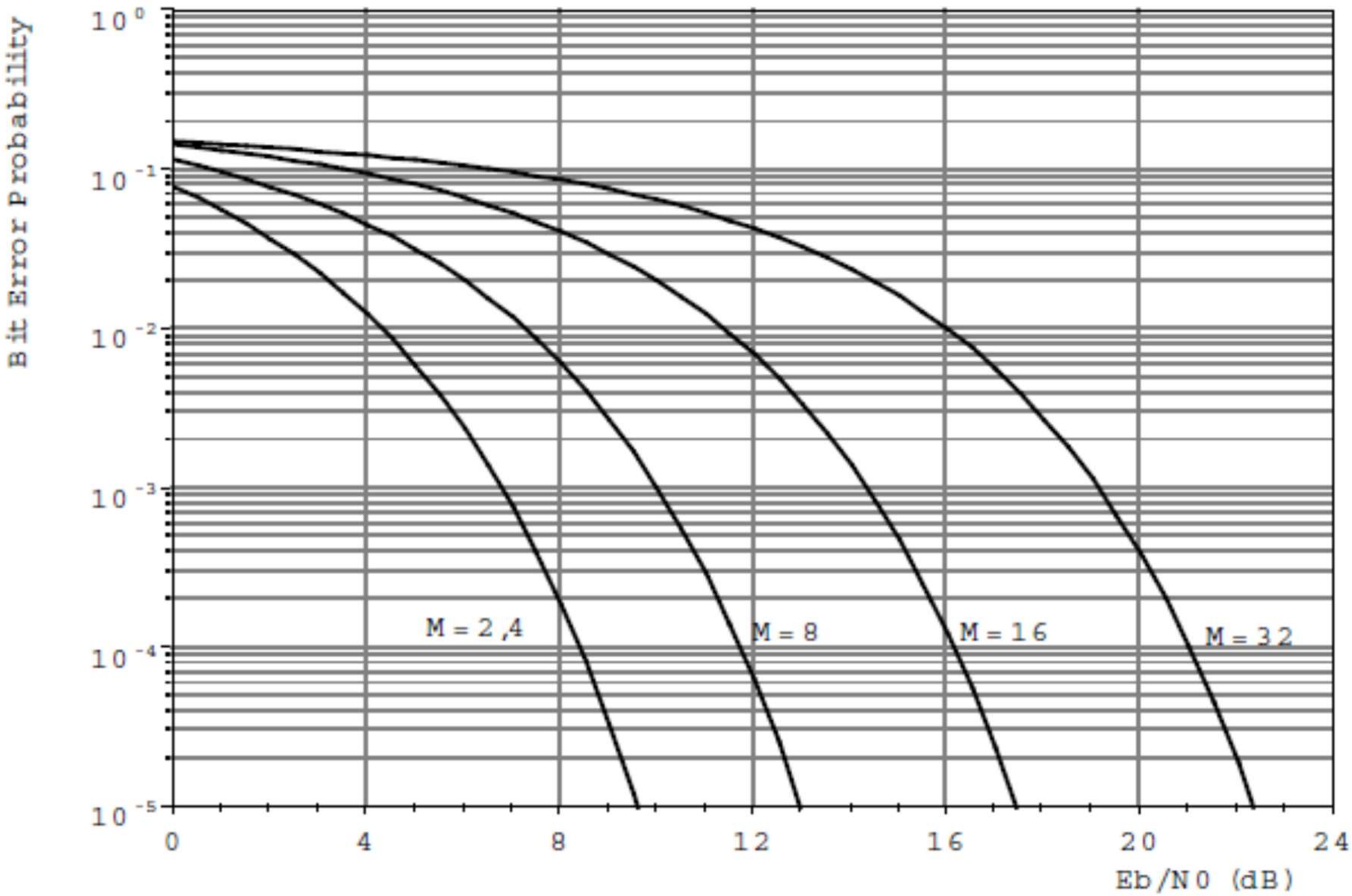
Symbol error probability $P_{e,s,QPSK}$ (or SER)

$$P_{e,s,QPSK} = 1 - P_{c,I} \cdot P_{c,Q} = 1 - \left(1 - \frac{1}{2}\text{erfc}\sqrt{\frac{E_s}{2 \cdot N_0}}\right)^2 = \text{erfc}\sqrt{\frac{E_s}{2 \cdot N_0}} - \frac{1}{4}\left(\text{erfc}\sqrt{\frac{E_s}{2 \cdot N_0}}\right)^2$$

Bit error probability (BER) at high SNR + Gray mapping (one symbol error ⇒ one bit error)

$$P_{e,b,QPSK} \cong \frac{1}{2}\text{erfc}\sqrt{\frac{2 \cdot E_b}{2 \cdot N_0}} - \underbrace{\frac{1}{4}\left(\text{erfc}\sqrt{\frac{E_s}{2 \cdot N_0}}\right)^2}_{\approx 0} = \frac{1}{2}\text{erfc}\sqrt{\frac{E_b}{N_0}} = P_{e,BPSK}$$

M-ary Phase Modulation



M-ary Quadrature Amplitude Modulation – MQAM

Waveforms:

$$s_i(t) = a_k \cdot \varphi_1(t) + q_l \cdot \varphi_2(t)$$

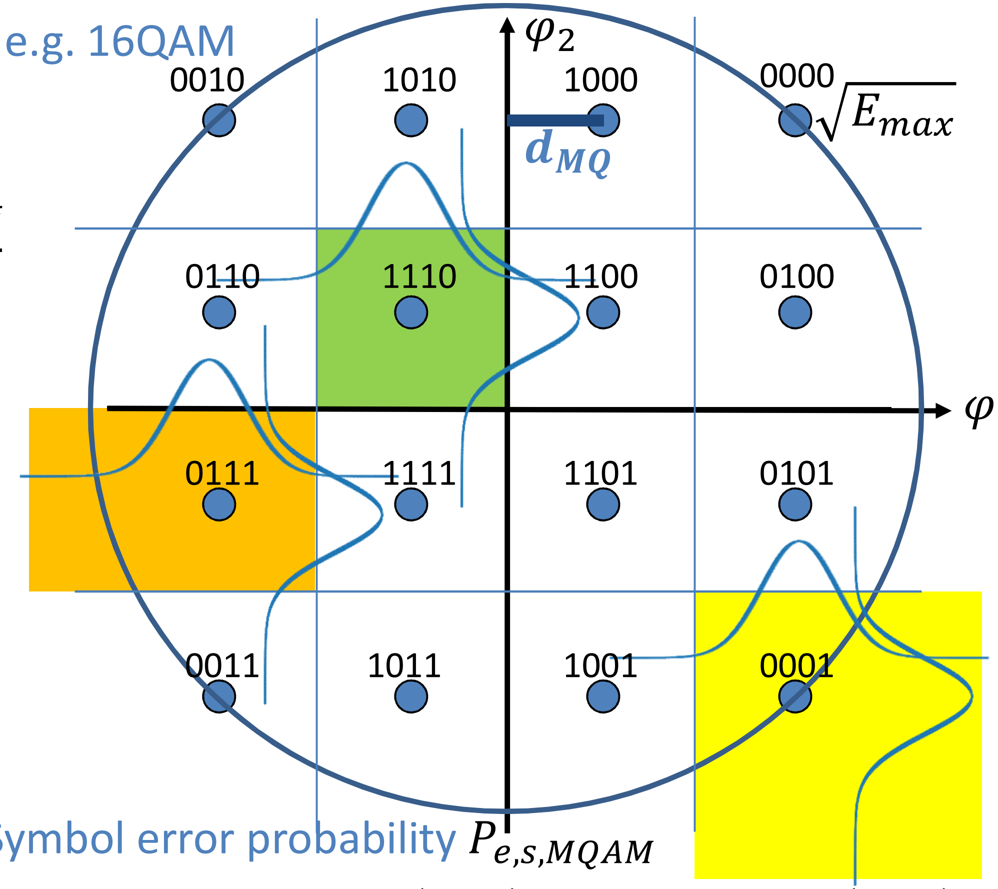
$$a_i, q_i = \sqrt{\frac{E_{max}}{2}} \cdot \frac{2i-1}{\sqrt{M}-1} \quad i = \pm 1, \pm 2, \ldots, \pm \frac{\sqrt{M}}{2}$$

Different probability of error for symbols

$E_i \leq E_{max}$

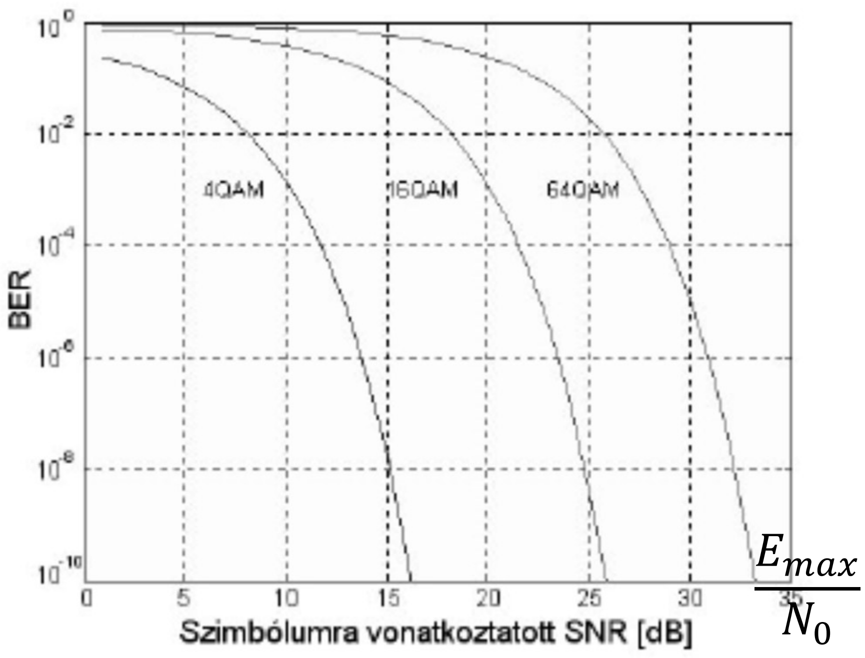
Distance to decision border $\boldsymbol{d_{MQ}} = \frac{\sqrt{E_{max}}}{\sqrt{2}(\sqrt{M}-1)}$

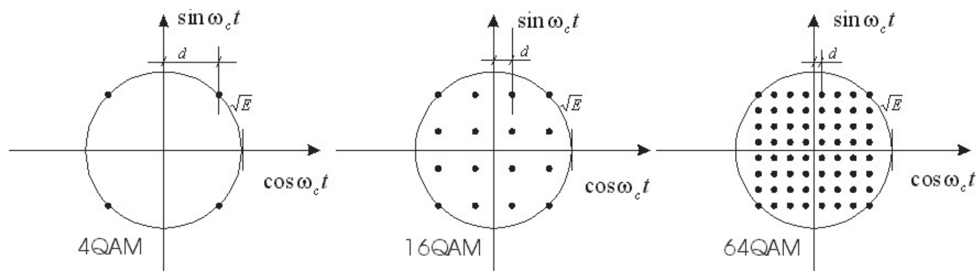
e.g. 16QAM



Symbol error probability $P_{e,s,MQAM}$

$$P_{E_{QAM}} = 2\left(1-\frac{1}{\sqrt{M}}\right)\frac{1}{2}erfc\left(\frac{d}{\sqrt{N_0}}\right) - \left(1-\frac{1}{\sqrt{M}}\right)^2 erfc^2\left(\frac{d}{\sqrt{N_0}}\right)$$





MPSK versus MQAM

BPSK	MPSK	MQAM

Symbol energy:

$$E_b = P_{BP} \cdot T_b \qquad E_s = P_{MP} \cdot T_b \cdot ld\ M \qquad E_{max} = P_{MQ} \cdot T_b \cdot ld\ M$$

Euclidean Distance to decision border:

$$\boldsymbol{d_{BP}} = \sqrt{E_b} \qquad \boldsymbol{d_{MP}} = \sqrt{E_s}\sin\frac{\pi}{M} \qquad \boldsymbol{d_{MQ}} = \frac{\sqrt{E_{max}}}{\sqrt{2}(\sqrt{M}-1)}$$

Same Euclidean Distance to decision border ↔ Approximately the same probability of error

$$\frac{\boldsymbol{d_{MP}}}{\boldsymbol{d_{BP}}} = 1 = \frac{\sqrt{E_s}\sin\frac{\pi}{M}}{\sqrt{E_b}} = \frac{\sqrt{P_{MP} \cdot T_b \cdot ld\ M}\sin\frac{\pi}{M}}{\sqrt{P_{BP} \cdot T_b}} \leftrightarrow \frac{P_{MP}}{P_{BP}} = \frac{1}{\left(\sin\frac{\pi}{M}\right)^2 \cdot ld\ M}$$

$$\frac{\boldsymbol{d_{MQ}}}{\boldsymbol{d_{BP}}} = 1 = \frac{\frac{\sqrt{E_{max}}}{\sqrt{2}(\sqrt{M}-1)}}{\sqrt{E_b}} = \frac{\sqrt{P_{MQ} \cdot T_b \cdot ld\ M}}{\sqrt{2}(\sqrt{M}-1)\sqrt{P_{BP} \cdot T_b}} \leftrightarrow \frac{P_{MQ}}{P_{BP}} = \frac{2 \cdot (\sqrt{M}-1)^2}{ld\ M}$$

n=ld M bit	$M = 2^n$	$B_n \cdot T_b$	P_{MP}/P_{BP} [dB]	P_{MQ}/P_{BP} [dB]	Comments
1	2	1	0	0	
2	4	0,5	0	0	QPSK! B/2
3	8	0,33	3,57	x	
4	16	0,25	8,17	6,53	QAM>PSK
5	32	0,2	13,2	x	
6	64	0,17	18,4	12,1	6 dB for QAM

M-ary orthogonal Frequency Shift Keying – MFSK

Waveforms:

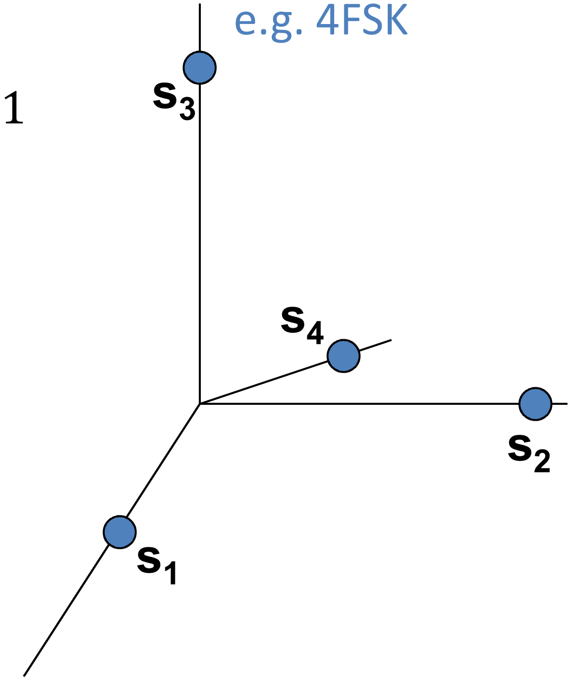
$$s_i(t) = \sqrt{\frac{2 \cdot E_S}{T_S}} \cos[(\omega_c + i \cdot \delta\omega) \cdot t + \varphi_i] \quad i = 0, \dots, M-1$$

Frequency deviation: $\delta\omega$

φ_i could be any

Signal set is orthogonal if $\delta\omega = 2\pi/T_s$

e.g. 4FSK



Symbol error probability

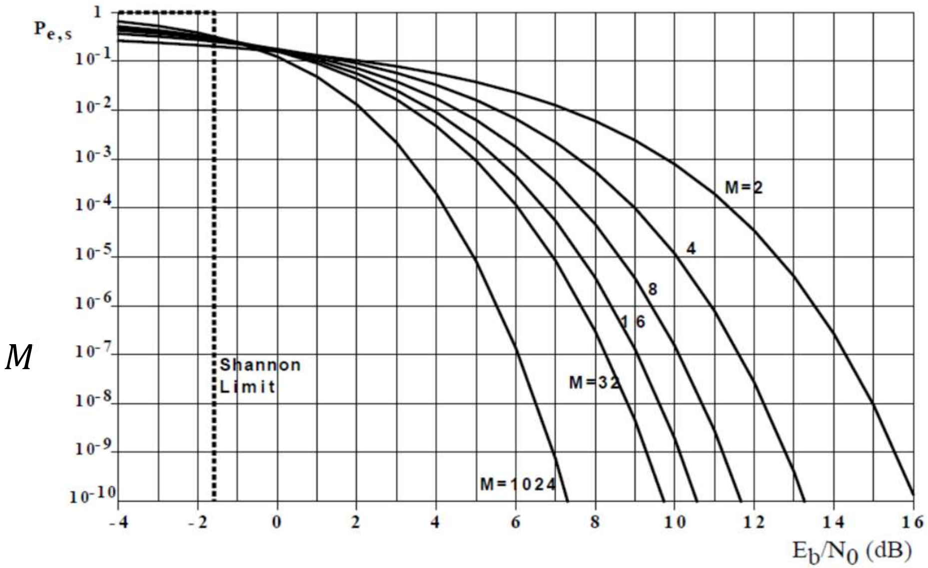
$$P_E = \frac{M-1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi\left(u - \sqrt{\frac{2E}{N_0}}\right) \Phi^{M-2}(u) e^{-u^2/2} du$$

where $\Phi(u)$ is cumulative distribution of G($\mu = 0, \sigma = 1$) standard Gaussian function

$$\Phi(u) = \frac{1}{2\pi} \int_{-\infty}^{u} e^{-x^2/2}\, dx$$

Upper bound at high SNR

$$P_E \leq \begin{cases} 1, & E/N_0 \leq \ln M \\ exp\left\{-\left(\sqrt{E/N_0} - \sqrt{\ln M}\right)^2\right\}, & \ln M < E/N_0 \leq 4 \cdot \ln M \\ exp\{-(E/2N_0 - \ln M)\}, & E/N_0 > 4 \cdot ln \end{cases}$$



M-ary Biorthogonal Frequency Shift Keying

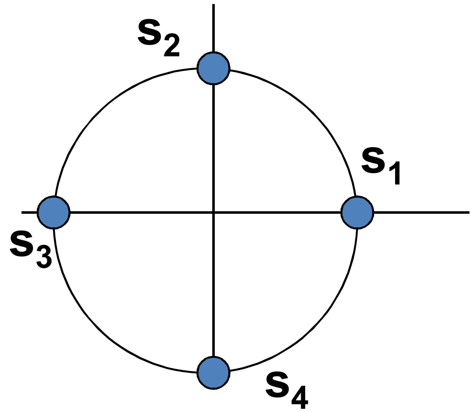
M -ary *biorthogonal* signal constellation is obtained from a set of M/2 orthogonal signals (such as M/2-*FSK*)

Waveforms:

$\pm s_i(t) \;\; i = 0, \ldots, M/2$

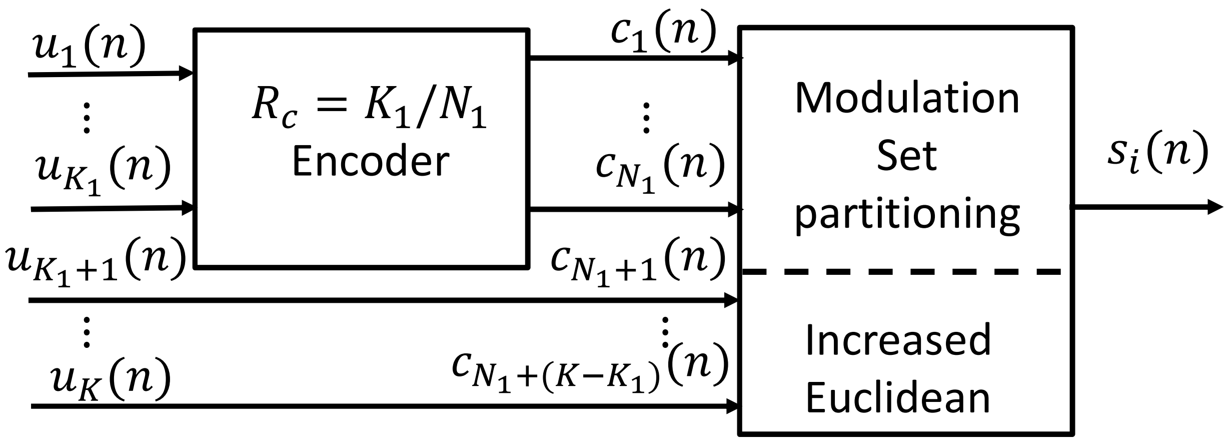
Symbol error performance is similar as by QPSK

e.g. biorthogonal 4FSK



Coded Modulation

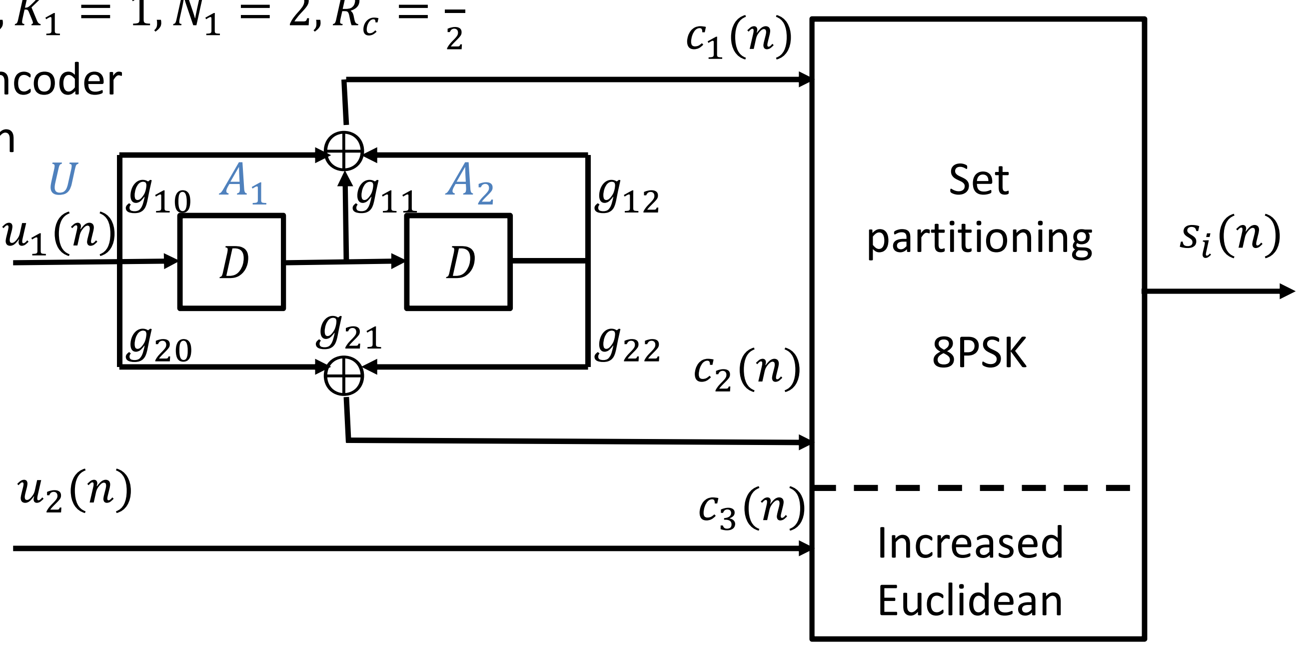
Goal: Increasing Euclidean distance using set partitioning. Redundant modulation symbols.
Advantage at Demodulation: Protection of subset selection by encoding, increased distance inside the subset.

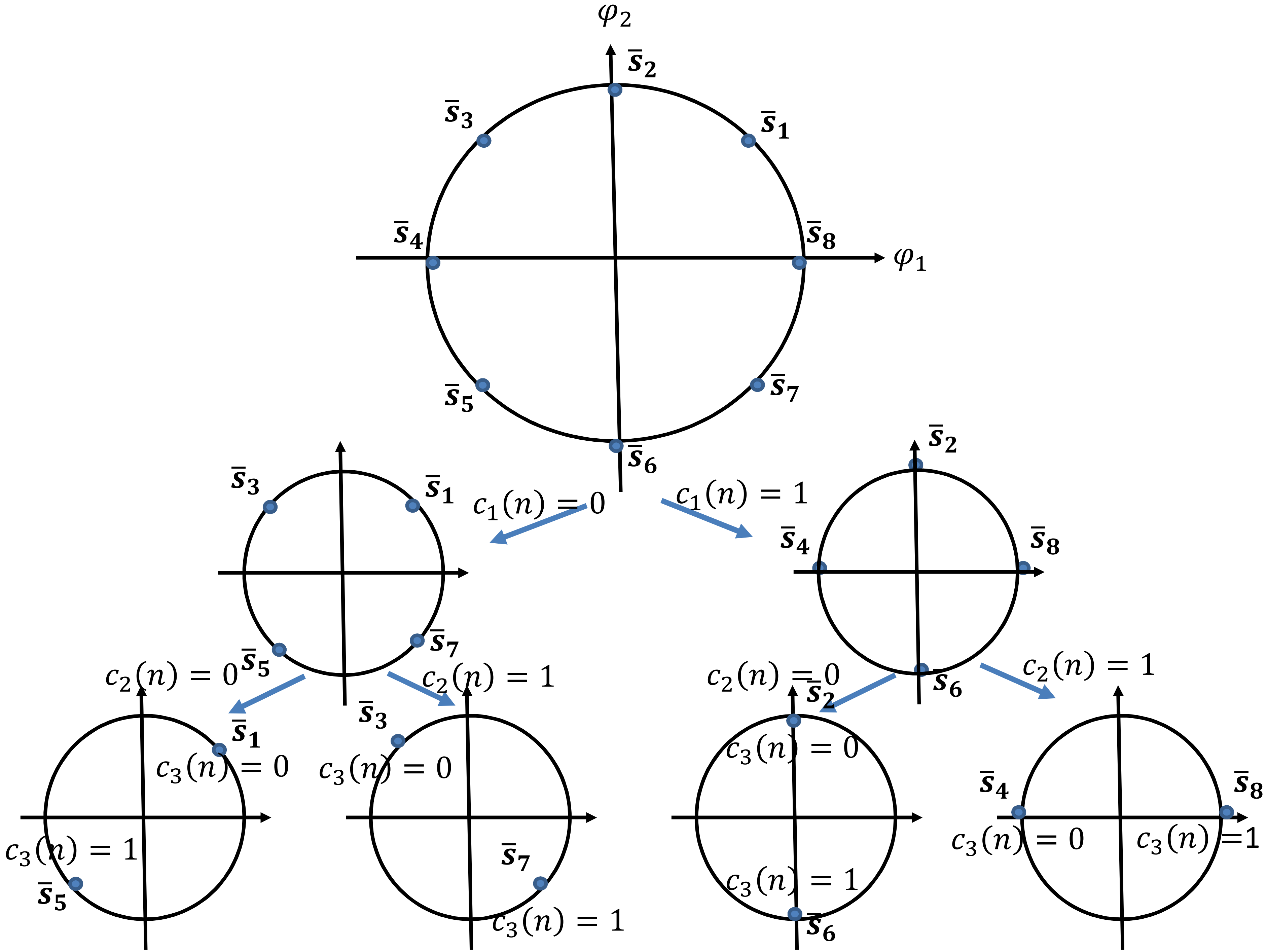


Example: $K = 2, K_1 = 1, N_1 = 2, R_c = \frac{1}{2}$

Convolutional Encoder

8PSK Modulation





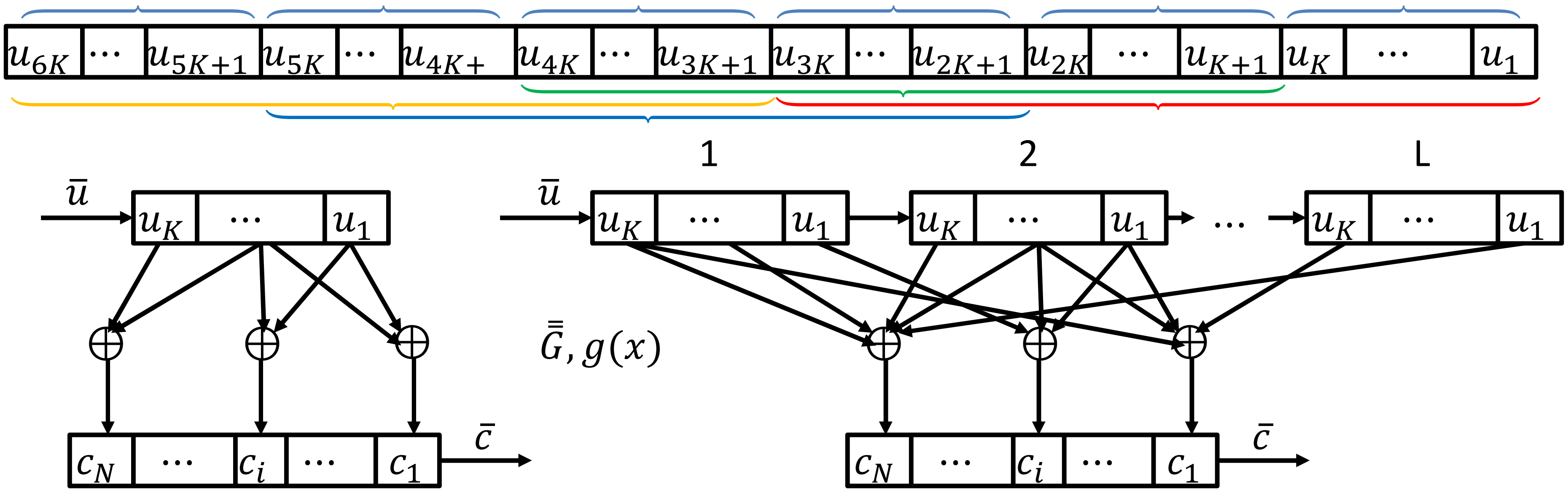
φ_2
$\bar{s}_2$
$\bar{s}_3$
$\bar{s}_1$
$\bar{s}_4$
$\bar{s}_8$
φ_1
$\bar{s}_5$
$\bar{s}_7$
$\bar{s}_6$
$c_1(n) = 0$
$c_1(n) = 1$
$c_2(n) = 0$
$c_2(n) = 1$
$c_2(n) = 0$
$c_2(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$

Convolutional coding, trellis coding, Viterbi (de)coding

Block coding versus convolutional coding: both linear, but convolutional coding have memory

Block diagram of code generation with shift registers:

Block coding (fixed window) Convolutional coding (moving window)



Parameters of convolutional codes: N, K, L, q, $R_c = K/N$

- N: Length of the code vector
- K: Length of the message vector
- L: Constrain length
- R_c: Coding rate
- q: Size of GF(q), mostly binary, q=2

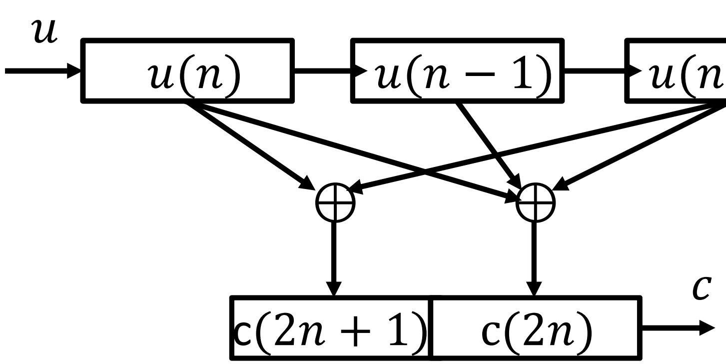
Convolutional coding

Some features:

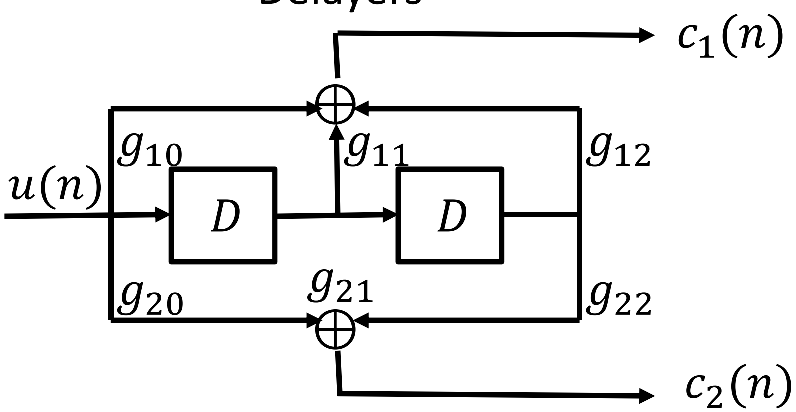
- Linear codes, because remains in the code space
- Generator polynomials are non-trivial, computer search for appropriate polynomials are necessary; there are no algorithms for defining the polynomials
- There are systematic, but usually non-systematic codes
- Possibility for „soft decoding”

Example: (N=2, K=1, q=2, L=3); Realization with

Registers



Delayers



$$\bar{g}_1 = [1\ 1\ \ 1] = 7_{octal}$$

$$\bar{g}_2 = [1\ 0\ \ 1] = 5_{octal}$$

$$\bar{c}_1 = \bar{u} * \bar{g}_1, \bar{c}_2 = \bar{u} * \bar{g}_2$$

Convolutional coding

Generates the code vectors as discrete convolution of the message vector with the generator vectors:

$$\bar{c}(n) = [c_1(n), c_2\,(n), \cdots, c_N(n)],$$

$$c_i(n) = \left(\sum_{j=0}^{L-1} u(n-j)\cdot g_{ij}\right) mod\ q$$

Unilateral Z-transform: converts a discrete-time signal x(n), which is a sequence of real or complex numbers, into a complex frequency-domain representation X(z).

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} x_n \cdot z^{-n}$$

In our case discrete –time sequence (i.e. u(n), c(n), etc.) of symbols (i.e. GF(q) elements) into frequency domain. Let use the notation: $D = z^{-1}$

Transformation of message sequence u(n) into message polynomial $U(D)$

$$U(D) = \mathcal{Z}\{u(n)\} = \sum_{j=0}^{\infty} u_j \cdot D^j$$

Transformation of generator sequence g_i(n) into generator polynomial $G_i(D)$

$$G_i(D) = \sum_{j=0}^{L-1} g_{ij} \cdot D^j$$

Convolutional coding

A code polynomial $C_i(D) = U(D) \cdot G_i(D)$

The code vector polynomial is then the vector of code polynomials

$$\bar{C}(D) = [C_1(D), C_2(D), \cdots, C_N(D)] = U(D) \cdot \bar{G}^T(D)$$

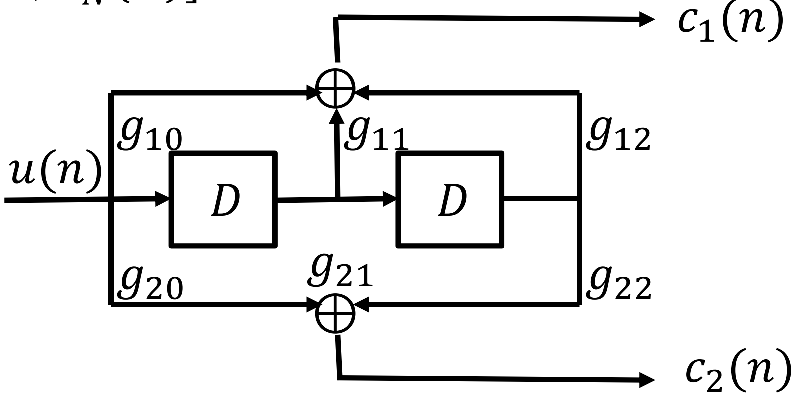
Where $\bar{G}^T(D)$ is the transpose of the vector of generator polynomials

$$\bar{G}(D) = [G_1(D), G_2(D), \cdots, G_N(D)]$$

Example: (N=2, K=1, q=2, L=3);

$G_1(D) = g_{10} + g_{11} \cdot D^1 + g_{12} \cdot D^2 = 1 + D^1 + D^2$

$G_2(D) = g_{20} + g_{21} \cdot D^1 + g_{22} \cdot D^2 = 1 + D^2$



Let the message: u(n)= 1 0 0 1 1 ....

Z-transform: $U(D) = 1 + D^3 + D^4 + \cdots$

Code polynomials:

$$C_1(D) = U(D) \cdot G_1(D) = 1 + D^3 + D^4 + D + D^4 + D^5 + D^2 + D^5 + D^6 + \cdots$$
$$= 1 + D + D^2 + D^3 + D^6 + \cdots$$

$$C_2(D) = U(D) \cdot G_2(D) = 1 + D^3 + D^4 + D^2 + D^5 + D^6 + \cdots$$
$$= 1 + D^2 + D^3 + D^4 + D^5 + D^6 + \cdots$$

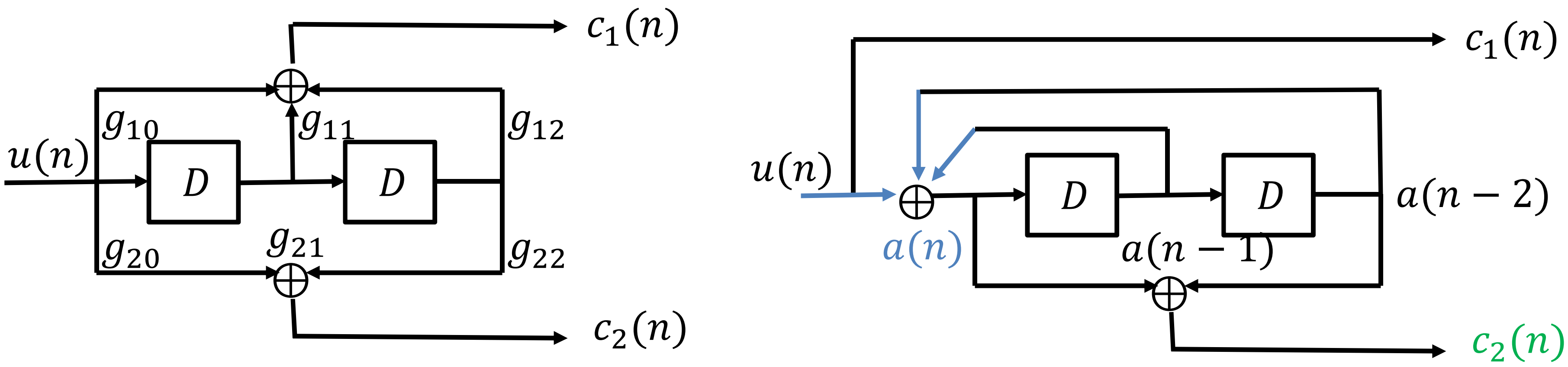
$\bar{C}(D) = [C_1(D), C_2(D)]$

The code sequence: c(n)=$\overbrace{11}^{D^0}\ \overbrace{10}^{D^1}\ \overbrace{11}^{D^2}\ \overbrace{11}^{D^3}\ \overbrace{01}^{D^4}\ \overbrace{01}^{D^5}\ \overbrace{11}^{D^6} \ldots$

Systematic versus non-systematic codes

- NSC: **N**on-recursive, non-**S**ystematic, **C**onvolutional codes
  The current state of the coder is independent from the structure
  Usual case except turbo coding, where RSC is applied
- RSC: **R**ecursive, **S**ystematic **C**onvolutional codes
  The current state of the coder depends from the structure

NSC structure　　Derive an RSC from an NSC　　RSC structure



$G_1(D) \Rightarrow \tilde{G}_1(D) = 1$ Systematic

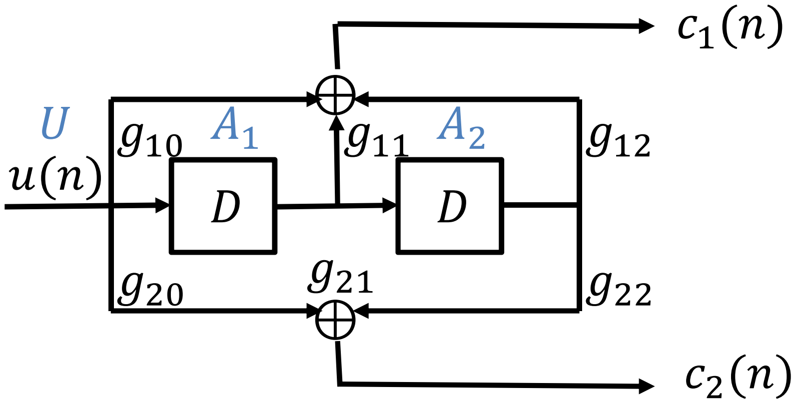
$G_2(D) \Rightarrow \tilde{G}_2(D) = \frac{G_2(D)}{G_1(D)} = \frac{1+D^2}{1+D^1+D^2}$　　$A(D) = \frac{U(D)}{1+D^1+D^2} \Rightarrow U(D) = A(D) \cdot (1 + D^1 + D^2)$

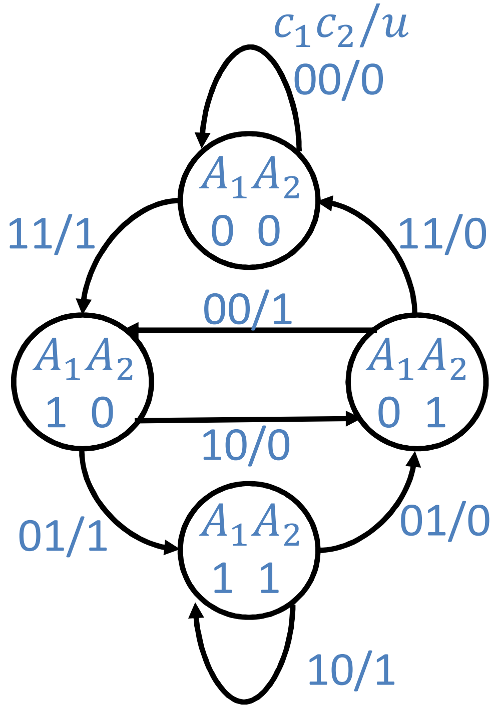
$\tilde{C}_1(D) = U(D) \cdot \tilde{G}_1(D) = U(D)$　　$\Rightarrow u(n) = a(n) + a(n-1) + a(n-2)$

$\tilde{C}_2(D) = U(D) \cdot \tilde{G}_2(D) = \underbrace{\frac{U(D)}{G_1(D)}}_{A(D)} G_2(D)$　　$\Rightarrow a(n) = u(n) + a(n-1) + a(n-2)$

Graphical representation

Describing the time behavior of the encoding process: State diagram versus trellis diagram

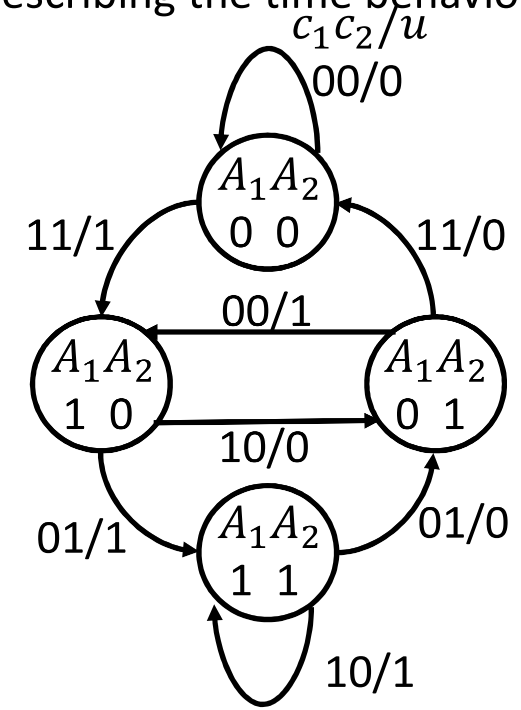




U	A_1	A_2	c_1	c_2
0	0 0	0 0	0	0
1	0 1	0 0	1	1
0	0 0	1 0	1	1
1	0 1	1 0	0	0
0	1 0	0 1	1	0
1	1 1	0 1	0	1
0	1 0	1 1	0	1
1	1 1	1 1	1	0

Graphical representation

Describing the time behavior of the encoding process: State diagram versus trellis diagram



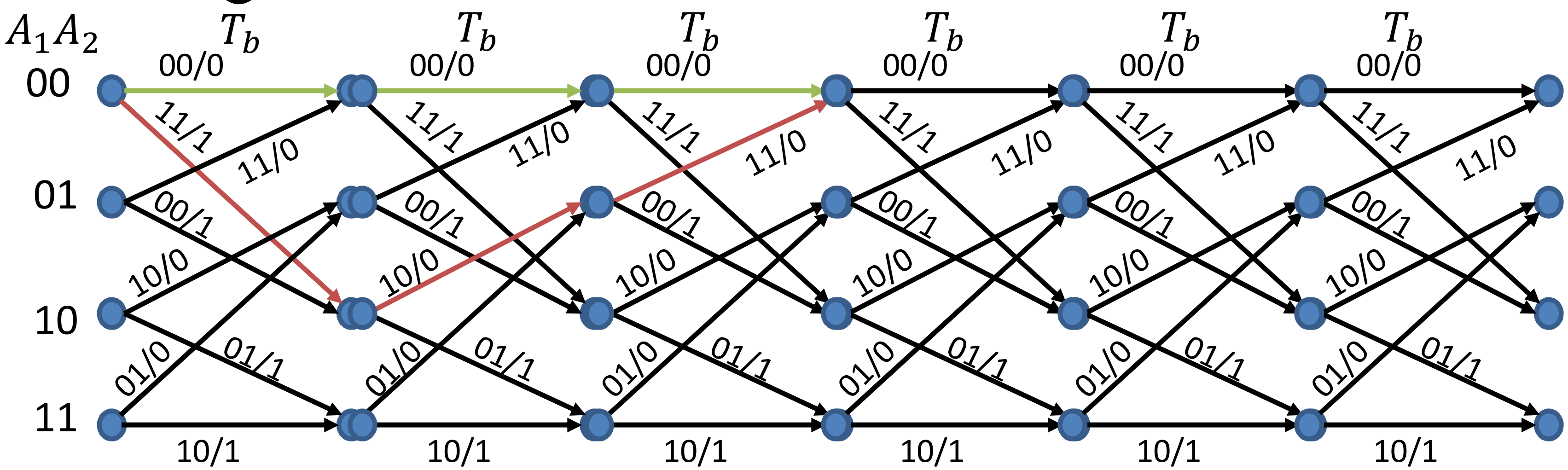
- Each path corresponds to a code symbol sequence
- Hamming distance separate paths

Minimum Hamming distance d_{free} (=5 in this case)

00 00 00

11 10 11 d=5

Decoding: Estimate the most probable path



Decoding criterions

- MAP – Maximum A-posteriori Probability criteria

$$\hat{c}(n) = arg \max_{c(n)}\{p\big(c(n) \mid v(n)\big)\}\,;\; with\ a - posteriori\ p\big(c(n) \mid v(n)\big)$$

Or equivalently

$$p(\hat{c}(n) \mid v(n)) \geq p\big(c(n) \mid v(n)\big) \qquad \forall\, c(n)$$

Multiplying with $p\big(v(n)\big)$

$$p(\hat{c}(n) \mid v(n)) \cdot p\big(v(n)\big) \geq p\big(c(n) \mid v(n)\big) \cdot p\big(v(n)\big) \qquad \forall\, c(n)$$

According to Bayes criteria

$$p(\hat{c}(n), v(n)) \geq p\big(c(n), v(n)\big) \qquad \forall\, c(n)$$

$$p\big(v(n) \mid \hat{c}(n)\big) \cdot p\big(\hat{c}(n)\big) \geq p\big(v(n) \mid c(n)\big) \cdot p\big(c(n)\big) \quad \forall\, c(n)$$

Therefore:

$$\hat{c}(n) = arg \max_{c(n)}\{p\big(v(n) \mid c(n)\big) \cdot p\big(c(n)\big)\};\; with\ a - priori\ p\big(c(n)\big)$$

- ML – Maximum Likelihood criteria

$$\hat{c}(n) = arg \max_{c(n)}\{p\big(v(n) \mid c(n)\big)\};\; without\ a - priori\ p\big(c(n)\big)$$

If $p\big(c(n)\big)$ uniform, then MAP ≡ ML

Decoding criterions

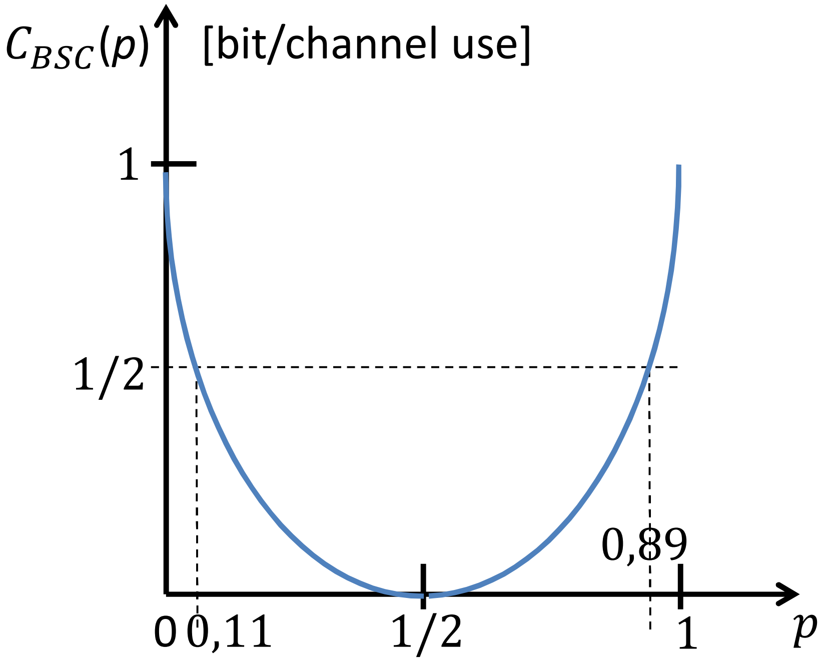
- Maximum Likelihood estimation over BSC with error probability p

$$\hat{c}(n) = arg\max_{c(n)}\{p(v(n) \mid c(n))\}$$

For vectors of length N

$$p(\bar{v} \mid \bar{c}) = (1-p)^{N-d(\bar{v},\bar{c})} \cdot p^{d(\bar{v},\bar{c})} = (1-p)^{N} \cdot \left(\frac{p}{1-p}\right)^{d(\bar{v},\bar{c})}$$

$$\hat{\bar{c}} = arg\max_{\bar{c}}\{p(\bar{v} \mid \bar{c})\} = arg\,min_{\bar{c}}\,\{d(\bar{v},\bar{c})\}$$



Decoding criterions

- Maximum Likelihood (ML) decision

In favor of the vector $\bar{c}$ from all the possible 2^N vectors (sequence) of length N according to

$$\hat{\bar{c}} = arg\,\max_{\bar{c}}\{p(\bar{v} \mid \bar{c})\} = arg\,\min_{\bar{c}}\ \{d(\bar{v}, \bar{c})\}$$

Number of required distance calculations: 2^N (exponentially increasing).

- Viterbi algorithm, Maximum Likelihood Sequence Estimation (MLSE)

At closing of alternative routes (starting from the same state and ending at the same state of the trellis) discarding path with higher Hamming distance to the received sequence of length N.
Number of required distance calculations: $\sim N$ (Linear increasing).

Example: (N=2, K=1, L=3, q=2), $G_1(D) = 1 + D^1 + D^2,\ G_2(D) = 1 + D^2$

00 00 00
11 10 11 d=5

11 11 01
00 01 10 d=5

indifferent

Suppose 00 00 00, decision will be wrong, if we receive 3, 4, or 5 errors, e.g. 10 10 10 -> 11 10 11

Probability of wrong decision at d=5 over BSC(p)

$$P_{d=5} = \sum_{i=3}^{5} \binom{5}{i} \cdot p^i \cdot (1-p)^{5-i}$$

Resulting error probability

- Convolutional coding, Viterbi decoding, BSC(p)

Upper bound of wrong decision probability at *d*

$$P_d = \begin{cases} \sum_{i=\frac{d+1}{2}}^{d} \binom{d}{i} \cdot p^i \cdot (1-p)^{d-i} \quad for\ d\ odd \\ \frac{1}{2} \cdot \binom{d}{d/2} \cdot p^{d/2} \cdot (1-p)^{d/2} + \sum_{i=\frac{d}{2}+1}^{d} \binom{d}{i} \cdot p^i \cdot (1-p)^{d-i} \quad for\ d\ even \end{cases}$$

Upper bound of resulting error probability at $N \to \infty$ (at the end of transmission)

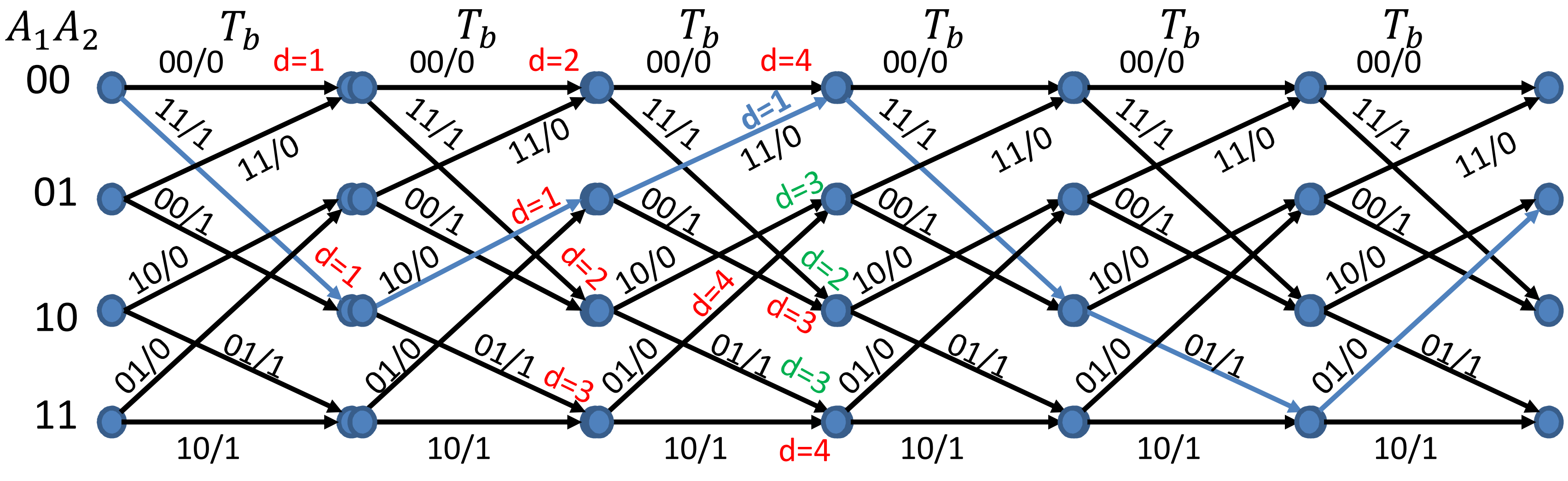
$$P_{e,BSC} < \sum_{d=\frac{d_{free}}{2}}^{\infty} a_d \cdot P_d$$

Where a_d is the number of closing of alternative routes (loops) with Hamming *d* of constructing paths.

For BPSK over AWGN with SNR $P_e < \sum_{d=d_{free}}^{\infty} erfc\sqrt{SNR \cdot R_c \cdot d}$

Example: Trellis diagram

- Example parameter of convolutional coding (N=2, K=1, L=3, q=2)
- Encoding the Message sequence $u(n)$ into Code vector sequence $\bar{c}(n)$
- Cumulate the Hamming distance from the Received vector sequence $\bar{v}(n)$ along all possible path
- Estimate the most probable path

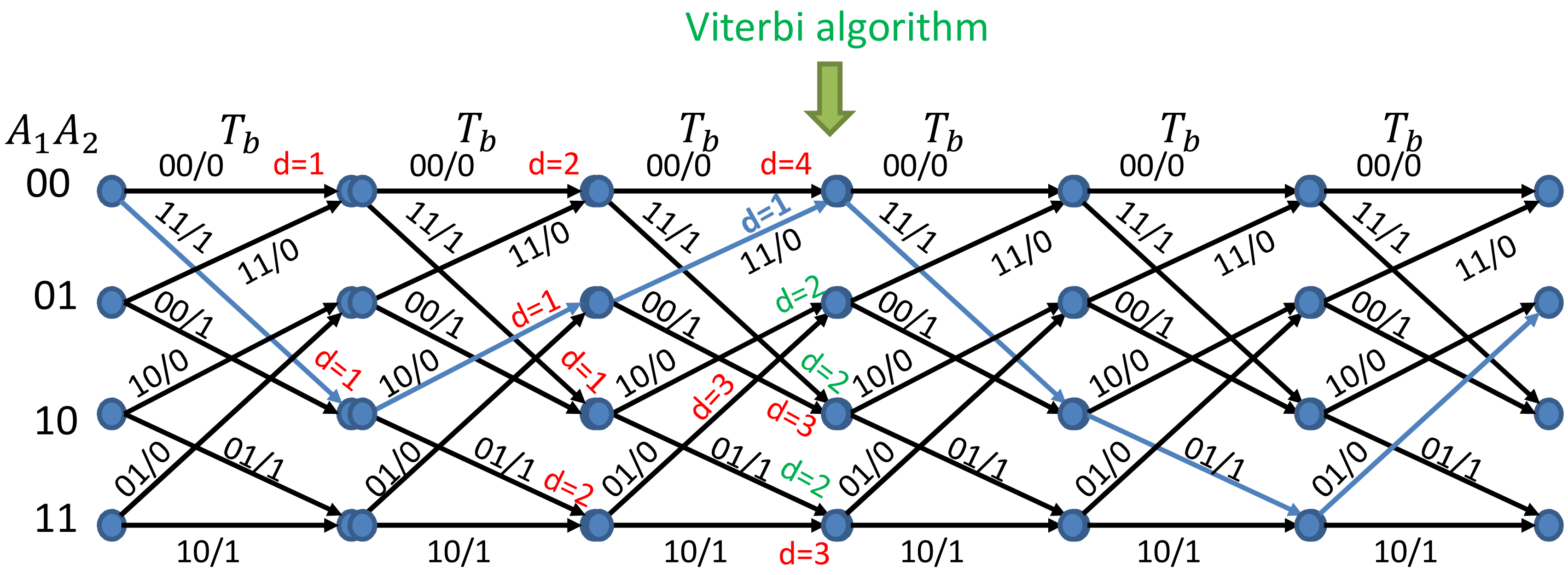


$u(n)$	1	0	0	1	1	0
$\bar{c}(n)$	11	10	11	11	01	01
$\bar{v}(n)$	10	10	11	11	01	01

Decision here

Puncturing

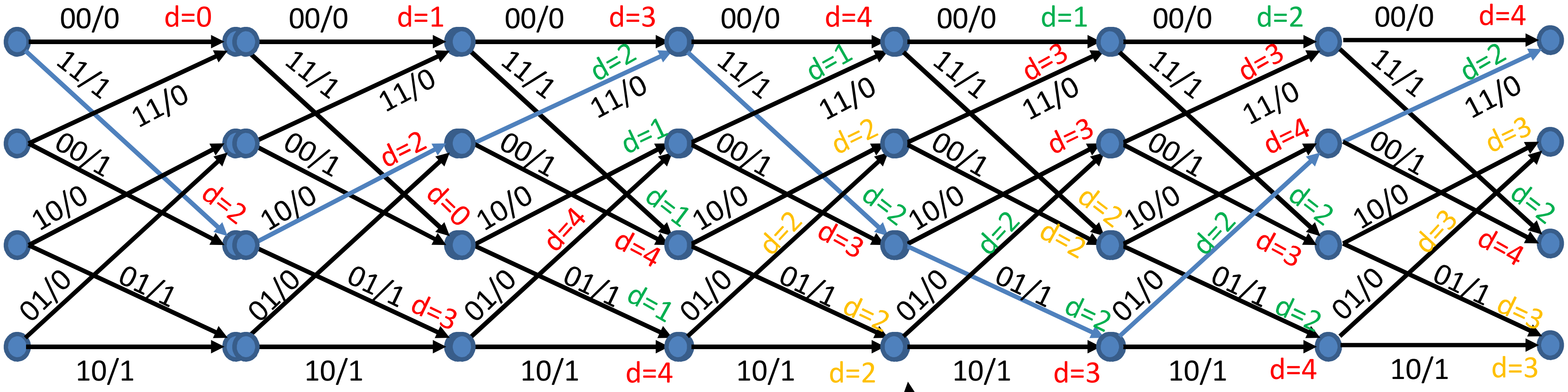
- Indifferent code symbols could be avoided by transmission
- Even other code symbols could be avoided adapting the coding rate R_c to the channel quality



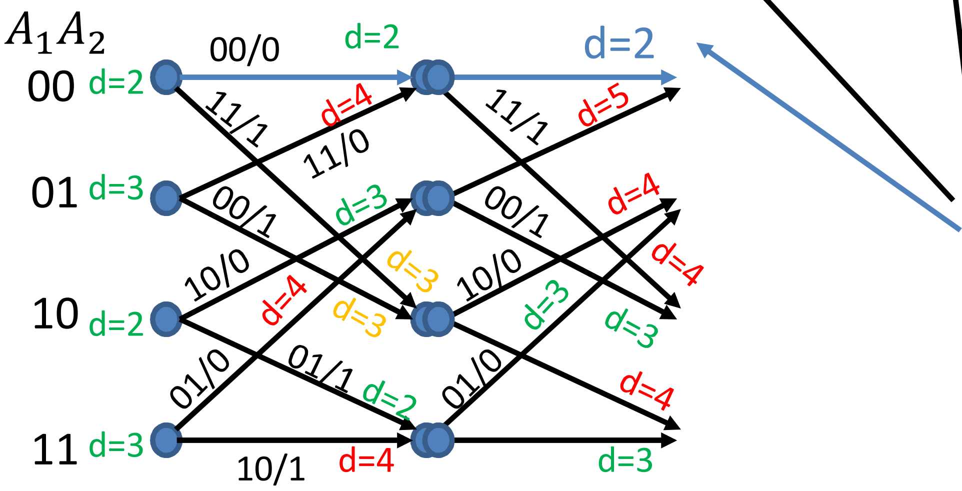
$u(n)$	1	0	0	1	1	0
$\bar{c}(n)$	11	1X	11	11	01	01
$\bar{v}(n)$	10	1X	11	11	01	01

Hamming (N=7, K=4, q=2) versus Punctured convolutional coding (N=2, K=1, L=3, q=2)

$R_c = 4/7 \cong 0{,}6$ $R_c = 3/5 = 0{,}6$



$u(n)$	1	0	0	1	1	0	0
$\bar{c}(n)$	11	1X	11	11	0X	01	11
$\bar{v}(n)$	00	1X	11	11	0X	01	11



$u(n)$	0	0
$\bar{c}(n)$	0X	00
$\bar{v}(n)$	0X	00

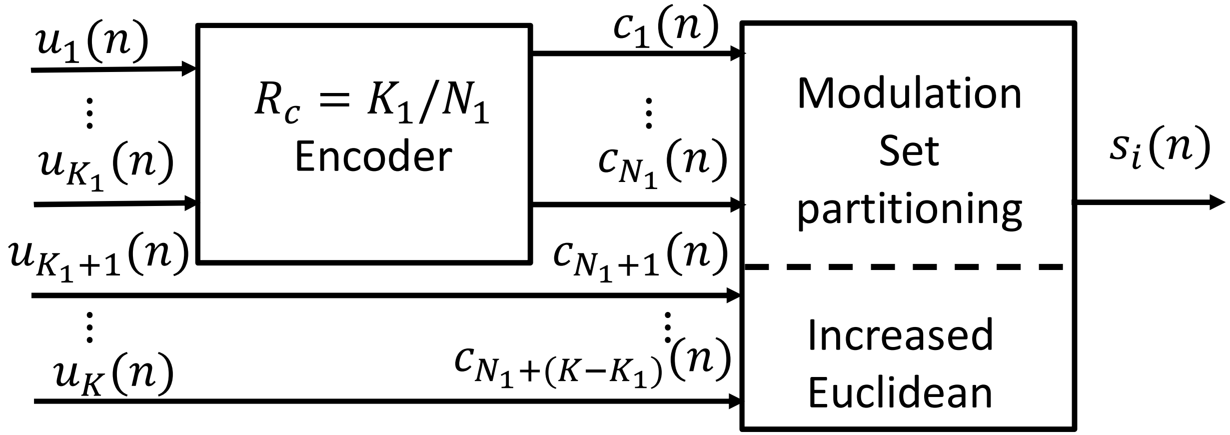
Decision when?

Optimally at the end of transmission

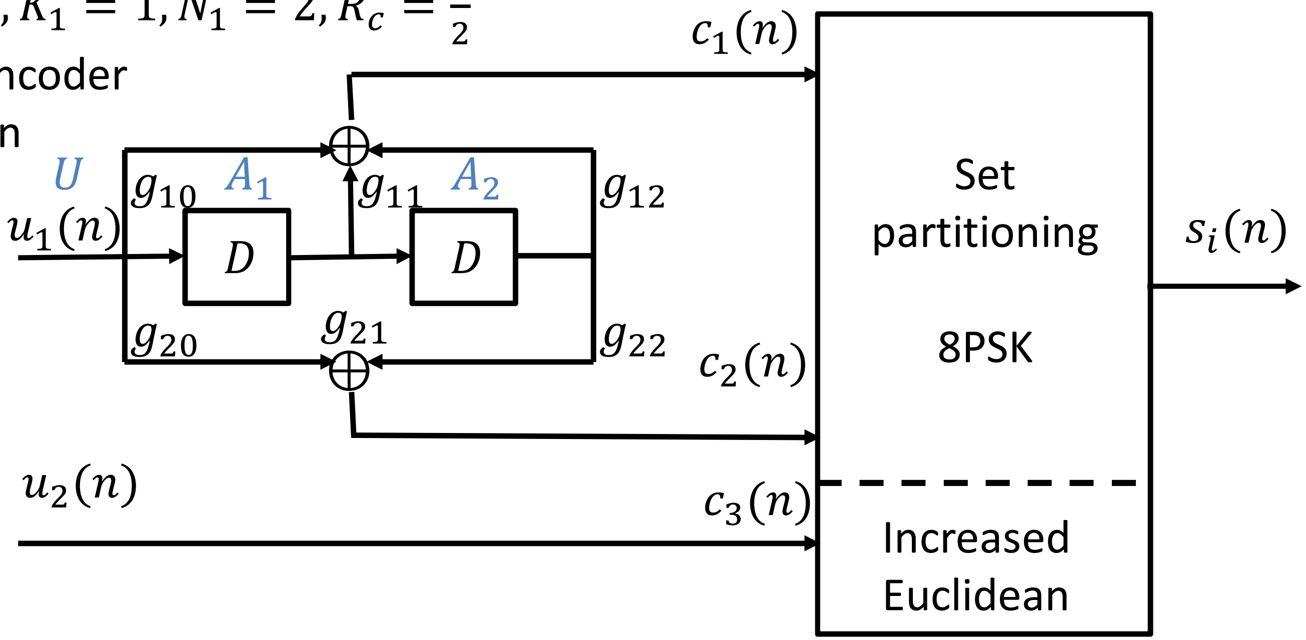
Corrected 2 errors at approximately the same R_c

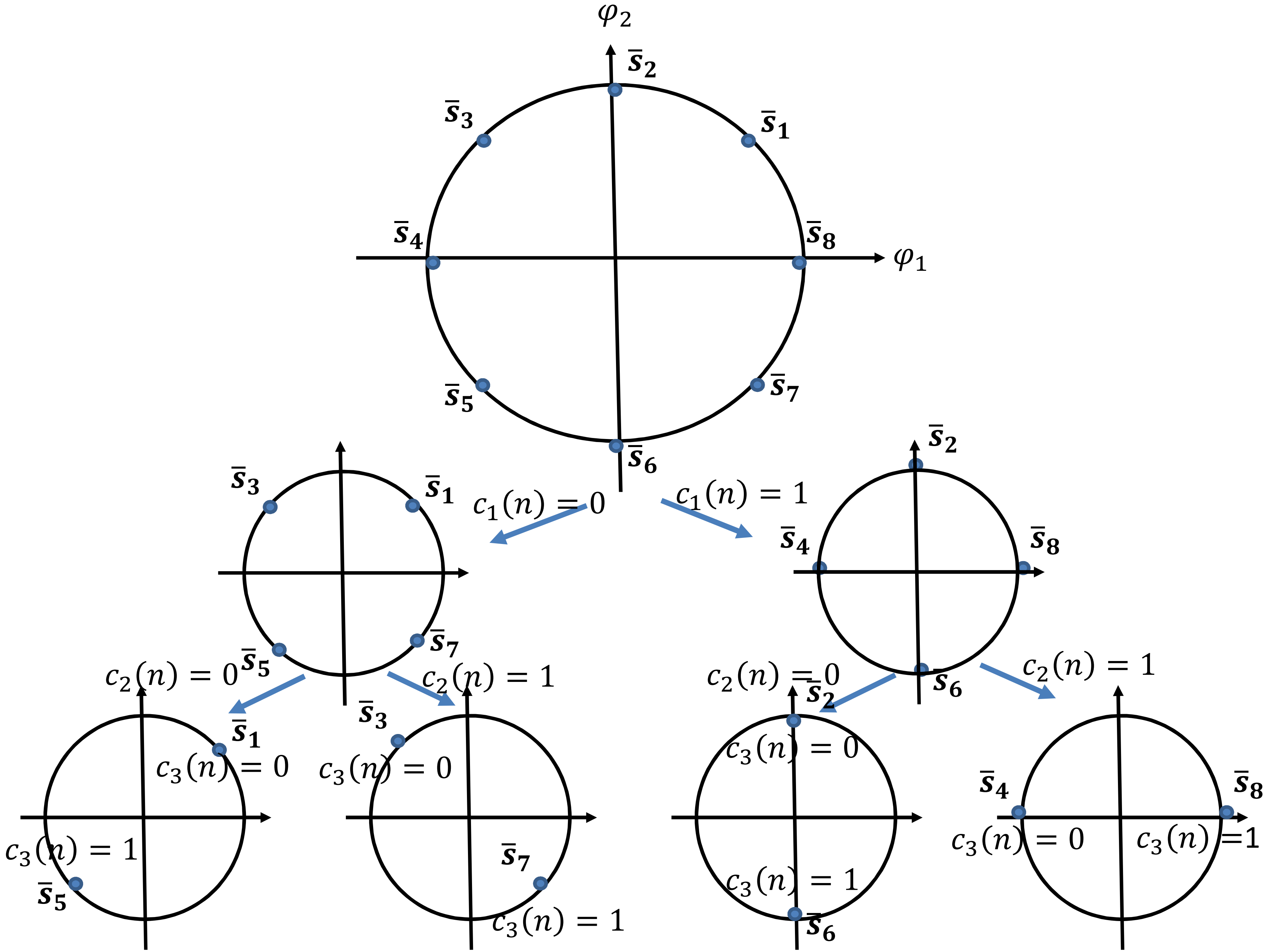
Coded Modulation

Goal: Increasing Euclidean distance using set partitioning. Redundant modulation symbols.
Advantage at Demodulation: Protection of subset selection by encoding, increased distance inside the subset.

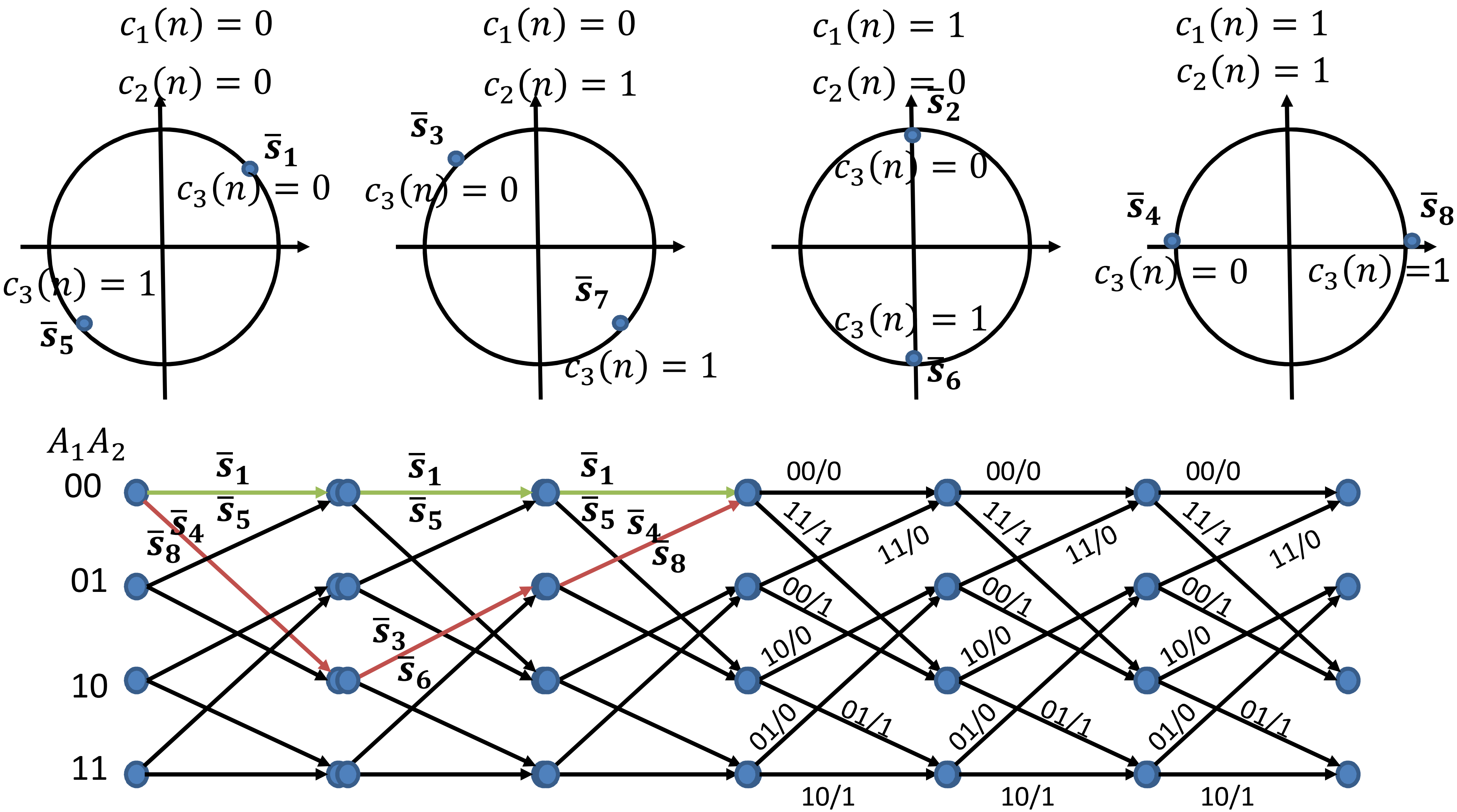


Example: $K = 2, K_1 = 1, N_1 = 2, R_c = \frac{1}{2}$
Convolutional Encoder
8PSK Modulation

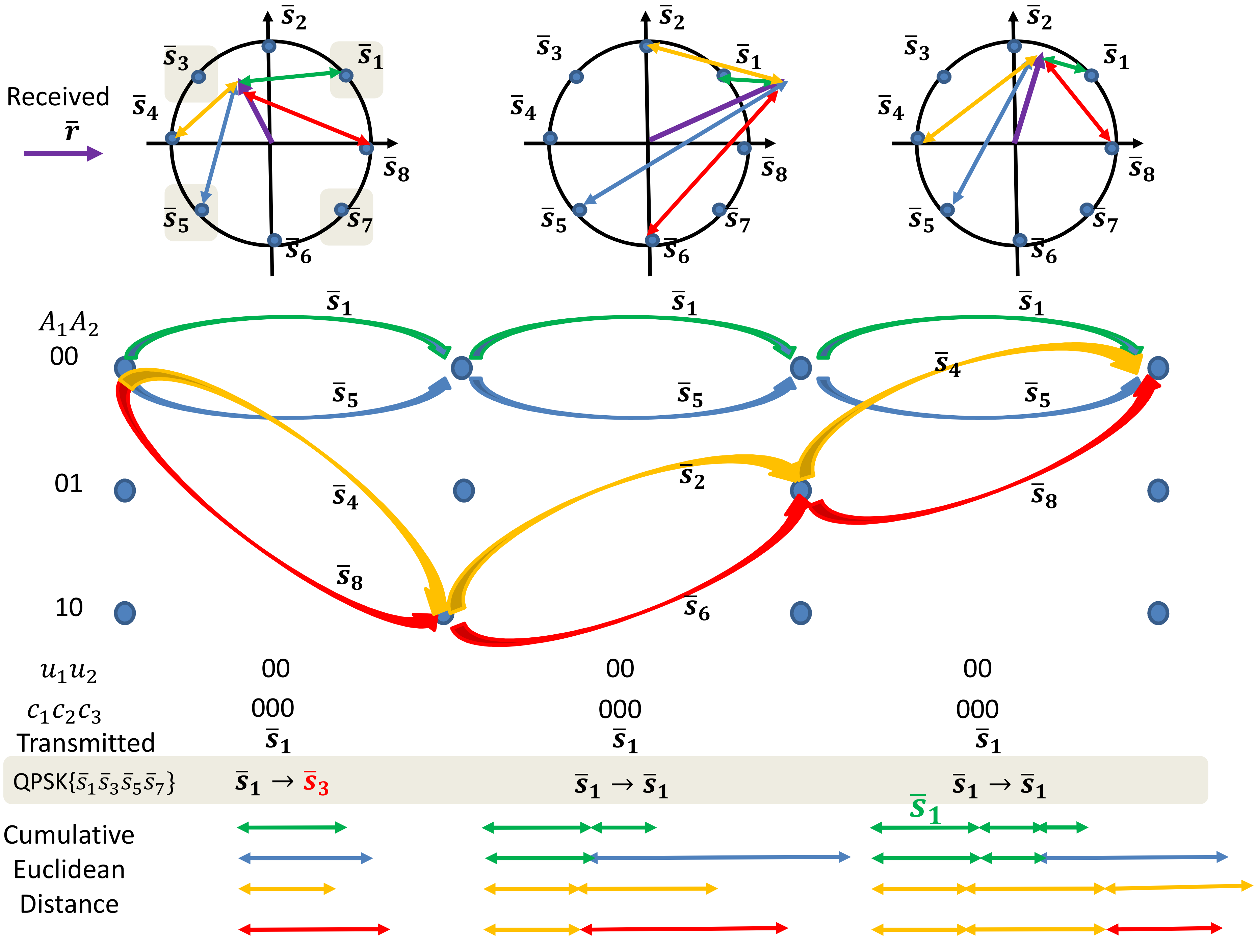




φ_2
$\bar{s}_2$
$\bar{s}_3$
$\bar{s}_1$
$\bar{s}_4$
$\bar{s}_8$
φ_1
$\bar{s}_5$
$\bar{s}_7$
$\bar{s}_6$
$c_1(n) = 0$
$c_1(n) = 1$
$c_2(n) = 0$
$c_2(n) = 1$
$c_2(n) = 0$
$c_2(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$
$c_3(n) = 0$
$c_3(n) = 1$



$c_1(n) = 0$
$c_2(n) = 0$
$\bar{s}_1$
$c_3(n) = 0$
$c_3(n) = 1$
$\bar{s}_5$
$c_1(n) = 0$
$c_2(n) = 1$
$\bar{s}_3$
$c_3(n) = 0$
$\bar{s}_7$
$c_3(n) = 1$
$c_1(n) = 1$
$c_2(n) = 0$
$\bar{s}_2$
$c_3(n) = 0$
$c_3(n) = 1$
$\bar{s}_6$
$c_1(n) = 1$
$c_2(n) = 1$
$\bar{s}_4$
$\bar{s}_8$
$c_3(n) = 0$
$c_3(n) = 1$
A_1A_2
00
01
10
11
$\bar{s}_1$ $\bar{s}_5$ $\bar{s}_4$ $\bar{s}_8$
$\bar{s}_1$ $\bar{s}_5$ $\bar{s}_3$ $\bar{s}_6$
$\bar{s}_1$ $\bar{s}_5$ $\bar{s}_4$ $\bar{s}_8$
00/0
11/1
11/0
00/1
10/0
01/0
01/1
10/1



Received
$\bar{r}$
$\bar{s}_1$ $\bar{s}_2$ $\bar{s}_3$ $\bar{s}_4$ $\bar{s}_5$ $\bar{s}_6$ $\bar{s}_7$ $\bar{s}_8$
A_1A_2
00
01
10
$\bar{s}_1$ $\bar{s}_5$ $\bar{s}_4$ $\bar{s}_8$ $\bar{s}_2$ $\bar{s}_6$
u_1u_2 00 00 00
$c_1c_2c_3$ 000 000 000
Transmitted $\bar{s}_1$ $\bar{s}_1$ $\bar{s}_1$
QPSK$\{\bar{s}_1\bar{s}_3\bar{s}_5\bar{s}_7\}$ $\bar{s}_1 \rightarrow \bar{s}_3$ $\bar{s}_1 \rightarrow \bar{s}_1$ $\bar{s}_1 \rightarrow \bar{s}_1$
$\bar{s}_1$
Cumulative
Euclidean
Distance