2013. 05. 22. Info, anal 2. PPZH2 MINTAMEGOLDÁS

1, a, $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}} \cdot x^n$; $a_n = \frac{(-3)^n}{\sqrt{n}}$;

$\sqrt[n]{|a_n|} = \frac{3}{\sqrt[n]{\sqrt{n}}} \longrightarrow 3 \Longrightarrow R = \frac{1}{3}$ ③

④ Végpontok: $x = +\frac{1}{3}$ esetén : $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ konv., mert Leibniz.

$x = -\frac{1}{3}$ esetén : $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}} \left(\frac{-1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$ $\left(\alpha = \frac{1}{2} \le 1\right)$

Tehát K.T. $= \left(-\frac{1}{3}, +\frac{1}{3}\right]$

b, $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$; $a_n = \frac{(-3)^n}{n^{3/2}}$; $\sqrt[n]{|a_n|} = \frac{3}{\sqrt[n]{n^{3/2}}} \longrightarrow 3 \Longrightarrow R = \frac{1}{3}$ ③

④ Végpontok: $x = \pm\frac{1}{3}$: $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^{3/2}} \left(\frac{\pm 1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{(\mp 1)^n}{n^{3/2}}$ konvergens, mert absz. konv, mert $\alpha = \frac{3}{2} > 1$.

Tehát K.T. $= \left[-\frac{1}{3}, +\frac{1}{3}\right]$

2, $\boxed{7}$ a, $f(x) = \frac{1}{2+6x} = \frac{1}{2} \cdot \frac{1}{1+3x} = \frac{1}{2} \sum_{k=0}^{\infty} (-3x)^k = \sum_{k=0}^{\infty} \frac{(-3)^k}{2} x^k$, ④

ha $|-3x| < 1$, azaz $|x| < \frac{1}{3}$; K.T. $= \left(-\frac{1}{3}, \frac{1}{3}\right)$ ③

$\boxed{8}$ b, $g(x) = \frac{1}{(2+6x)^2} = \frac{-1}{6} f'(x) = \frac{-1}{6} \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{(-3)^k}{2} x^k\right) = \frac{-1}{6} \sum_{k=1}^{\infty} \frac{(-3)^k}{2} k x^{k-1} =$ ②

$f'(x) = \frac{-6}{(2+6x)^2}$

$= \sum_{k=0}^{\infty} \frac{(-3)^k}{4} (k+1) x^k$ ③ $R = \frac{1}{3}$ (mint előbb) ①

$x = \pm\frac{1}{3}$-ban a sor tagjai nem tartanak 0-hoz $\Longrightarrow$ K.T. $= \left(-\frac{1}{3}, +\frac{1}{3}\right)$ ②

2,c,

$\boxed{8}$ $h(x) = \ln(2+6x) = \ln 2 + \int\limits_{t=0}^{x} 6 f(t)\,dt = \ln 2 + 6 \int\limits_{t=0}^{x} \left(\sum_{k=0}^{\infty} \frac{(-3)^k}{2} t^k \right) dx =$

$h'(x) = \frac{6}{2+6x} = 6 f(x)$

$= \ln 2 + 6 \sum_{k=0}^{\infty} \frac{(-3)^k}{2} \cdot \frac{x^{k+1}}{k+1} = \ln 2 + \sum_{k=1}^{\infty} (-1)^{k-1} 3^k \cdot \frac{1}{k} x^k$ (5p)

$R = \frac{1}{3}$ (nem változik)

$x = \frac{1}{3}$-ban: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot \cancel{3^k}}{k} \left(\cancel{\frac{1}{3}}\right)^k =$ konv (Leibniz)

$x = -\frac{1}{3}$-ban: $\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\, 3^k}{k} \left(\frac{-1}{3}\right)^k = \sum_{k=1}^{\infty} \frac{-1}{k} = -\infty$ (div.) (3p)

$\Longrightarrow$ K.T. $= \left(-\frac{1}{3}, +\frac{1}{3}\right]$

3, $\boxed{11}$ $\sum_{n=0}^{\infty} (2n+1) x^{2n} = f(x)$

Tagonként integrálva:

$F(x) = \int\limits_0^{x} f(t)\,dt = \int\limits_0^{x} \left(\sum_{n=0}^{\infty} (2n+1) t^{2n} \right) dt = \sum_{n=0}^{\infty} (2n+1) \cdot \int\limits_0^{x} t^{2n}\,dt =$ (3)

$= \sum_{n=0}^{\infty} \cancel{(2n+1)} \frac{x^{2n+1}}{\cancel{2n+1}} = x \sum_{n=0}^{\infty} (x^2)^n = \frac{x}{1-x^2}$ (3), ha $|x^2| < 1$,

$R = 1$ (2)

Tehát $f(x) = F'(x) = \left(\frac{x}{1-x^2} \right)' = \frac{1-x^2 - x(-2x)}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2}$ (2)

$R = 1$ (nem változik) (1)

4, a, $\boxed{4}$

$$f(x)=\cos(x^2)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\,x^{4k} \quad ③ \quad ; \quad K.T. = \mathbb{R}. \quad ①$$

b, $\boxed{10}$

$$\int_0^1 \cos(x^2)\,dx = \int_0^1\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\,x^{4k}\right)dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left[\frac{x^{4k+1}}{4k+1}\right]_0^1 =$$

$$= \sum_{k=0}^{\infty}\underbrace{\frac{(-1)^k}{(2k)!}\cdot\frac{1}{4k+1}}_{a_k} \quad ⑤ \quad ; \quad \text{Leibniz sor}$$

$$\int_0^1 T_8(x)\,dx = \sum_{k=0}^{2}\frac{(-1)^k}{(2k)!}\,\frac{1}{4k+1} = \underline{1-\frac{1}{2!\cdot 5}+\frac{1}{4!\cdot 9}} = \underline{1-\frac{1}{10}+\frac{1}{24\cdot 9}} \quad ③$$

$$H \le |a_3| = \underline{\underline{\frac{1}{6!\cdot 13}}} \quad ②$$

5,

$$f(x,y)=\begin{cases} x\sin\left(\frac{1}{x^2+y^2}\right), & \text{ha } (x,y)\neq(0,0) \\ 0, & \text{ha } (x,y)=(0,0)\end{cases}$$

a, $\boxed{4}$

$$\lim_{(x,y)\to(0,0)}\Big(\underset{\downarrow\atop 0}{x}\,\underbrace{\sin\left(\frac{1}{x^2+y^2}\right)}_{\text{korlátos}}\Big) = 0 = f(0,0),$$ tehát f folytonos az origóban! ④

b, $\boxed{5}$ $(x,y)\neq(0,0)$

$$f'_x(x,y) = \sin\left(\frac{1}{x^2+y^2}\right) + x\cos\left(\frac{1}{x^2+y^2}\right)\cdot\frac{-2x}{(x^2+y^2)^2} \quad ③$$

$$f'_y(x,y) = x\cos\left(\frac{1}{x^2+y^2}\right)\cdot\frac{-2y}{(x^2+y^2)^2} \quad ②$$

c, $\boxed{9}$

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{1}{x}\left(x\sin\left(\frac{1}{x^2}\right)-0\right) = \lim_{x\to 0}\sin\left(\frac{1}{x^2}\right) = \nexists \quad ③$$

mert $x_k=\frac{1}{\sqrt{k\pi}}$ esetén $\sin\left(\frac{1}{x_k^2}\right)=0$, és $x_k=\frac{1}{\sqrt{\frac{\pi}{2}+2k\pi}}$ esetén $\sin\left(\frac{1}{x_k^2}\right)=1$ ③

$$f'_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{x\to 0}\frac{0-0}{y} = 0 \quad ③$$

6, $f(x,y) = \ln(3x^4y^2+1)$; $P_0(-1,1)$

a, $\boxed{9}$ $f'_x(x,y) = \dfrac{12x^3y^2}{3x^4y^2+1}$ ②

$f'_y(x,y) = \dfrac{6x^4y}{3x^4y^2+1}$ ②

$\operatorname{grad} f(P_0) = \begin{bmatrix} f'_x(P_0) \\ f'_y(P_0) \end{bmatrix} = \begin{bmatrix} -3 \\ 3/2 \end{bmatrix}$ ②

Érintő sík általában: $z - z_0 = f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$

Most: $f(-1,1) = \ln 4$; ① $\underline{z - \ln 4 = -3(x+1) + \frac{3}{2}(y-1)}$ ②

b, $\boxed{6}$ $\underline{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$; $\|\underline{v}\| = \sqrt{9+16} = 5$; $\underline{e} = \dfrac{\underline{v}}{\|\underline{v}\|} = \dfrac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ ②

$\left.\dfrac{df}{d\underline{e}}\right|_{P_0} = \underline{e} \cdot \operatorname{grad} f(P_0) = \dfrac{3}{5}\cdot(-3) + \dfrac{-4}{5}\cdot\dfrac{3}{2} = \dfrac{-9}{5} - \dfrac{6}{5} = \underline{\underline{-3}}$ ② ②

c, $\boxed{5}$ Iránymenti derivált $\dfrac{\operatorname{grad} f(P_0)}{\|\operatorname{grad} f(P_0)\|} = c \cdot \begin{bmatrix} -3 \\ 3/2 \end{bmatrix}$ ③ irányban maximális. Értéke: $\|\operatorname{grad} f(P_0)\| = \sqrt{(-3)^2 + \left(\frac{3}{2}\right)^2} = \dfrac{\sqrt{36+9}}{2} = \underline{\underline{\dfrac{3}{2}\cdot\sqrt{5}}}$ ②

7, $\boxed{10}$ $f(x,y) = g(3xy + y^2)$

$f'_x = g'(3xy+y^2)\cdot 3y$ ① ; $f'_y = g'(3xy+y^2)(3x+2y)$ ①

$f''_{xx} = g''(3xy+y^2)\cdot 9y^2$; ②

$f''_{xy} = f''_{yx}$ ② $= g''(3xy+y^2)\cdot 3y\cdot(3x+2y) + 3\cdot g'(3xy+y^2)$ ②

$f''_{yy} = g''(3xy+y^2)(3x+2y)^2 + 2g'(3xy+y^2)$ ②

8, $\boxed{10}$ $f(x) = (1+x)^{1/3}$; $T_3(x) = \sum_{k=0}^{3} \binom{1/3}{k} x^k = \binom{1/3}{0} + \binom{1/3}{1}x + \binom{1/3}{2}x^2 + \binom{1/3}{3}x^3 =$

$= 1 + \dfrac{1}{3}x + \dfrac{(1/3)(-2/3)}{2!}x^2 + \dfrac{(1/3)(-2/3)(-5/3)}{3!}x^3$