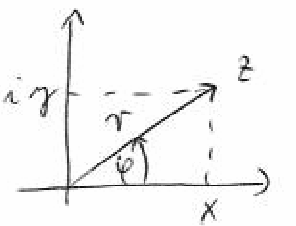
1, a,
$\boxed{6}$

$$z = \underbrace{x + iy}_{\text{algebrai}} \; ① = \underbrace{r e^{i\varphi}}_{\text{exp.}} \; ① = \underbrace{r(\cos\varphi + i \sin\varphi)}_{\text{trig. alak}} \; ①$$



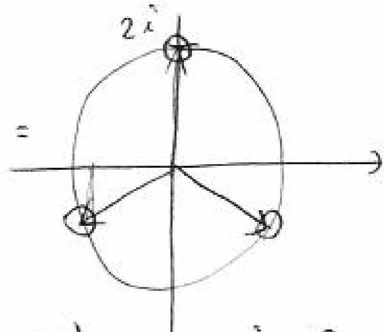
$$\left.\begin{array}{l} r = \sqrt{x^2+y^2} \; ; \; x = r\cos\varphi \; ; \; y = r\sin\varphi \\ \operatorname{tg}\varphi = \frac{y}{x} \; ; \; \operatorname{ctg}\varphi = \frac{x}{y} \end{array}\right\} ②$$

b,
$\boxed{6}$

$$z^3 = \left(r e^{i\varphi}\right)^3 = -8i = 8 \cdot e^{\frac{3\pi}{2} i} \implies r = \sqrt[3]{8} = 2 \quad ①$$

$$\varphi_k = \frac{\pi}{2} i + \frac{2\pi}{3} \cdot k \; ; \quad k = 0, 1, 2 \quad ②$$

$$z_1 = r e^{i\varphi_0} = 2 \cdot e^{\frac{\pi}{2} i} = \underline{\underline{2i}} \quad ①$$



$$z_2 = r e^{i\varphi_1} = 2 e^{i\left(\frac{\pi}{2} + \frac{2\pi}{3}\right)} = 2\left(\cos(210^\circ) + i\sin(210^\circ)\right) =$$

$$= \underline{\underline{-\sqrt{3} - i}} \quad ①$$

$$z_3 = r e^{i\varphi_2} = 2 e^{i\left(\frac{\pi}{2} + \frac{4\pi}{3}\right)} = 2\left(\cos(330^\circ) + i\sin(330^\circ)\right) = \underline{\underline{\sqrt{3} - i}} \quad ①$$

2, a,
$\boxed{6}$

T.: $\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}$ ②

B.: $$\left(\frac{1}{f}\right)'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{f(x+h)} - \frac{1}{f(x)}\right) = \lim_{h\to 0} \frac{1}{h} \frac{f(x) - f(x+h)}{f(x+h) f(x)} =$$

$$= \lim_{h\to 0} (-1) \underbrace{\frac{f(x+h) - f(x)}{h}}_{\downarrow h \to 0 \atop f'(x)} \cdot \frac{1}{f(x) \cdot \underbrace{f(x+h)}_{\downarrow h\to 0 \atop f(x)}} = -\frac{f'(x)}{f^2(x)} \quad ④$$

b,
$\boxed{4}$

$$(x^x)' = \left(\left(e^{\ln x}\right)^x\right)' = \left(e^{x \ln x}\right)' \overset{②}{=} \underbrace{e^{x\ln x}}_{x^x} \cdot \underbrace{(x \ln x)'}_{x \cdot \frac{1}{x} + 1 \cdot \ln x} =$$

$$= \underline{\underline{x^x (1 + \ln x)}} \quad ②$$

3, $\boxed{8}$ $x^2+2x-8=(x+4)(x-2)$ (2)

$$\frac{x}{x^2+2x-8}=\frac{A}{x+4}+\frac{B}{x-2} \quad (2) \quad \Rightarrow x = A(x-2)+B(x+4)$$

$$\left.\begin{array}{l} x=2 \Rightarrow 2 = B\cdot(2+4) \Rightarrow B=\frac{1}{3} \\ x=-4 \Rightarrow -4 = A(-4-2) \Rightarrow A=\frac{2}{3} \end{array}\right\} (2)$$

$$\int \frac{x}{x^2+2x-8}\,dx = \frac{2}{3}\int\frac{1}{x+4}\,dx+\frac{1}{3}\int\frac{1}{x-2}\,dx = \underline{\underline{\frac{2}{3}\ln|x+4|+\frac{1}{3}\ln|x-2|+C}} \quad (2)$$

4, a, $\boxed{7}$ Legyen φ és ψ megoldás, tehát

$$\varphi^{(n)}(x)+a_{n-1}(x)\varphi^{(n-1)}(x)+\dots+a_2(x)\varphi''(x)+a_1(x)\varphi'(x)+a_0(x)\varphi(x)=0$$

$$\psi^{(n)}(x)+a_{n-1}(x)\psi^{(n-1)}(x)+\dots+a_2(x)\psi''(x)+a_1(x)\psi'(x)+a_0(x)\psi(x)=0$$

Az első egyenletet szorozzuk $\alpha\in\mathbb{R}$-el, a másodikat $\beta\in\mathbb{R}$-el, és összeadjuk a két egyenletet:

$$\alpha\varphi^{(n)}+\beta\psi^{(n)}+a_{n-1}(\alpha\varphi^{(n-1)}+\beta\psi^{(n-1)})+\dots+a_2(\alpha\varphi''+\beta\psi'')+a_1(\alpha\varphi'+\beta\psi')+a_0(\alpha\varphi+\beta\psi)=0$$

$$\frac{d^n}{dx^n}(\alpha\varphi+\beta\psi)+a_{n-1}\frac{d^{n-1}}{dx^{n-1}}(\alpha\varphi+\beta\psi)+\dots+a_2(\alpha\varphi+\beta\psi)''+a_1(\alpha\varphi+\beta\psi)'+a_0(\alpha\varphi+\beta\psi)=0$$

Ez azt jelenti, hogy az $\alpha\varphi(x)+\beta\psi(x)$ is megoldás, tehát a megoldások valóban lineáris teret alkotnak.

$\boxed{8}$ b, (H) $y''+y'=0 \Rightarrow \lambda^2+\lambda=\lambda(\lambda+1)=0 \Rightarrow \underline{y_{H,ált}(x)=C_1+C_2\cdot e^{-x}}$ (3)

Külső rezonancia van! $y_{i,p}(x)=(Ax+B)\cdot e^{-x}\cdot x=(Ax^2+Bx)e^{-x}$ (2)

$$y_{i,p}'(x)=(2Ax+B)e^{-x}-(Ax^2+Bx)e^{-x}=(-Ax^2+(2A-B)x+B)e^{-x}$$

$$+\quad y_{i,p}''(x)=(-2Ax+2A-B+Ax^2-2Ax+Bx-B)e^{-x}=(Ax^2+(-4A+B)x+2A-2B)e^{-x}$$

$$(\cancel{-Ax^2}+\underline{(2A-B)x}+B+\cancel{Ax^2}+\underline{(-4A+B)x}+2A-2B)e^{-x}=x\,e^{-x}$$

$-2A=1 \Rightarrow A=-\frac{1}{2}$ (1)

$2A-B=-1-B=0 \Rightarrow B=-1$ (1)

$\underline{y_{i,p}(x)=\left(-\frac{x^2}{2}-x\right)e^{-x}}$ (1)

$\underline{\underline{y_{i,ált}(x)=C_1+C_2e^{-x}+\left(-\frac{x^2}{2}-x\right)e^{-x}}}$

5, a, [5]

(3) (D): $\sum_{n=1}^{\infty} a_n$ Leibniz-típusú, ha az a_n tagok váltakozó előjelűek, $\lim\limits_{n\to\infty} a_n = 0$, és $|a_1| \geq |a_2| \geq \dots \geq |a_n| \geq |a_{n+1}| \geq \dots$

(2) T.: Minden Leibniz-típusú sor konvergens.

b, [5] $\sum\limits_{n=0}^{\infty} \underbrace{(-1)^n \ln\left(\frac{3n}{2n+1}\right)}_{a_n}$;

$\lim\limits_{n\to\infty} \ln\left(\frac{3n}{2n+1}\right) = \ln\left(\underbrace{\lim\limits_{n\to\infty} \frac{3n}{2n+1}}_{3/2}\right) = \ln\left(\frac{3}{2}\right)$, így $\lim\limits_{n\to\infty} a_n \neq 0$,

tehát a sor divergens. (Nem teljesül a konv. szükséges feltétel.)

6, a, [6]

$$f(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-x^2)^k = \underline{\underline{\sum_{k=0}^{\infty} (-1)^k \cdot x^{2k}}} \quad (4)$$

ha $|-x^2| < 1$, azaz ha $\boxed{-1 < x < +1}$ $\underline{\underline{R = 1}}$ (2)

b, [6]

$$g(x) = \operatorname{arctg} x = \operatorname{arctg} 0 + \int_0^x \frac{1}{1+t^2}\,dt \ (2) = \int_0^x \left(\sum_{k=0}^{\infty} (-1)^k t^{2k}\right) dt \ (1) =$$

$$= \sum_{k=0}^{\infty} (-1)^k \int_0^x t^{2k}\,dt = \underline{\underline{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}}} \ (2); \quad \underline{\underline{R = 1}} \ (1)$$ (nem változik)

7, $f(x,y) = y^3 + 2(x+y)^2 - 8x - 20y$

[12]

$f'_x(x,y) = 4(x+y) - 8 = 0$ (1) $\Rightarrow x + y = 2;\ y = 2 - x$

$f'_y(x,y) = 3y^2 + 4(x+y) - 20 = 0$ (1) $\Rightarrow 3(2-x)^2 + 4 \cdot 2 - 20 = 0$

$(2-x)^2 = 4 \Rightarrow 2 - x = \pm 2$

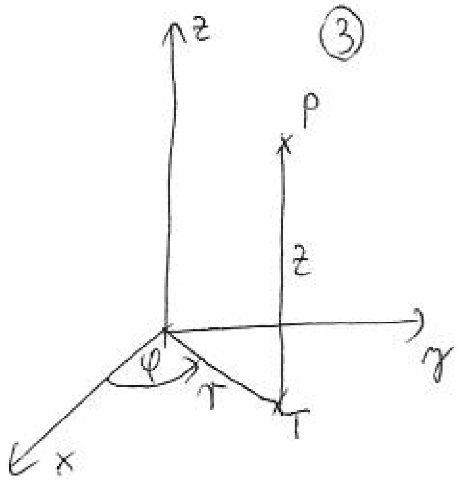
Vizsgálandó pontok: $(x_1, y_1) = (0, 2)$; $(x_2, y_2) = (4, -2)$ (4)

$$|\underline{\underline{H}}(x,y)| = \begin{vmatrix} 4 & 4 \\ 4 & 6y+4 \end{vmatrix} = 24y + 16 - 16 = 24y \ (2)$$

$|\underline{\underline{H}}(0,2)| = 48 > 0$; $f''_{xx} = 4 > 0 \Rightarrow (0,2)$-ben lok. min van (2)

$|\underline{\underline{H}}(4,-2)| = -48 < 0 \Rightarrow (4,-2)$-ben nyeregpont van (2)

8, $\boxed{10}$



3

$x = r\cos\varphi$ 1

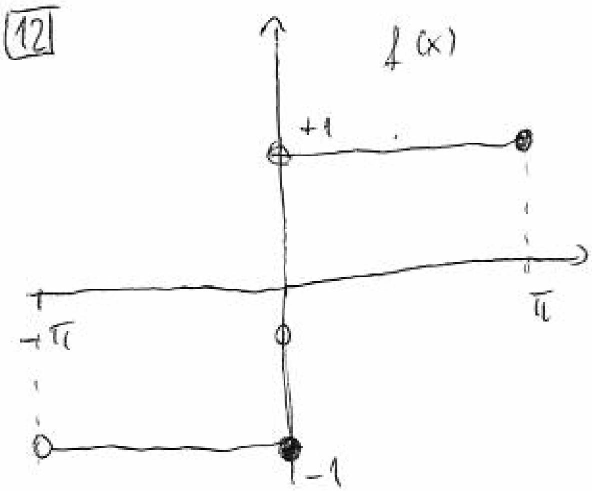
$y = r\sin\varphi$ 1

$z = z$ 1

$$J(r,\varphi,z) = \begin{vmatrix} x'_r & x'_\varphi & x'_z \\ y'_r & y'_\varphi & y'_z \\ z'_r & z'_\varphi & z'_z \end{vmatrix} = \quad 2$$

$$= \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\varphi + \sin^2\varphi) = \underline{\underline{r}} \quad 2$$

9, $\boxed{12}$



f páratlan $\Rightarrow a_n = 0$ 3 $(n = 0, 1, 2, \ldots)$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(x)\sin(nx)}_{\text{páros}}\,dx = \frac{2}{\pi}\int_0^{\pi} \underbrace{f(x)}_{1}\sin(nx)\,dx = \quad 2$$

$$= \frac{2}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{-2}{n\pi}\left((-1)^n - 1\right) =$$

$$= \begin{cases} \dfrac{4}{n\pi}, & \text{ha } n \text{ páratlan} \\ 0, & \text{ha } n \text{ páros} \end{cases} \quad 3$$

Tehát $$\underline{\underline{\Phi(x) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin\left((2k+1)x\right)}} \quad 2$$

$$\Phi(0) = \frac{f(0+0) + f(0-0)}{2} = \underline{\underline{0}}$$ (Dirichlet-tétel alapján) 2